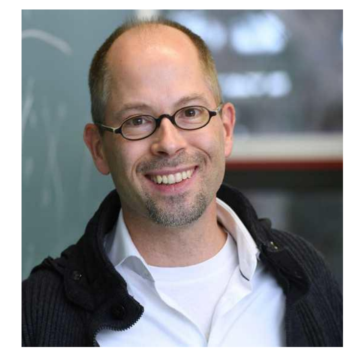
Gottesman-Kitaev-Preskill bosonic error correcting codes: a lattice perspective

Jonathan Conrad, Francesco Arzani, Jens Eisert



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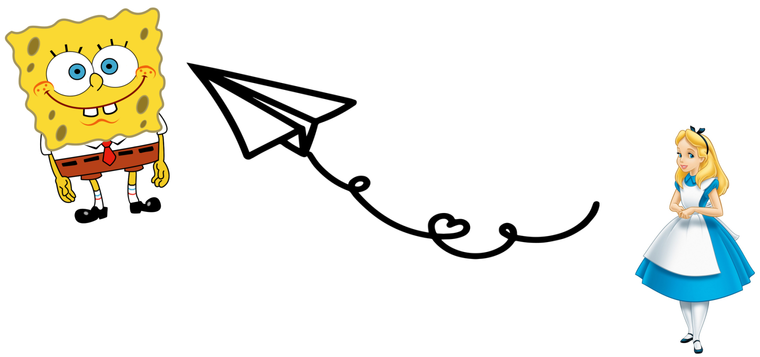
(Quantum) Error correction and harmonic oscillators



Information always encoded in phys. syst.

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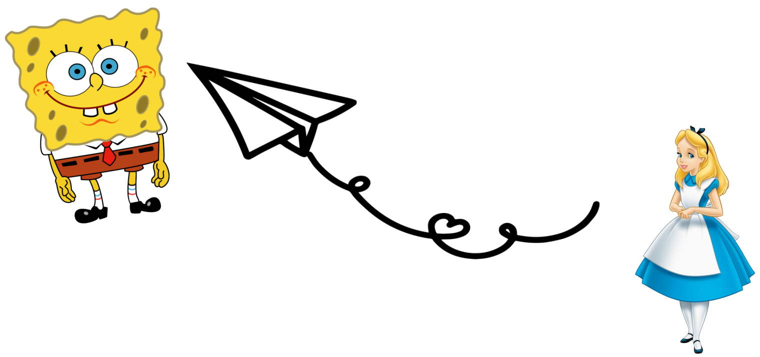

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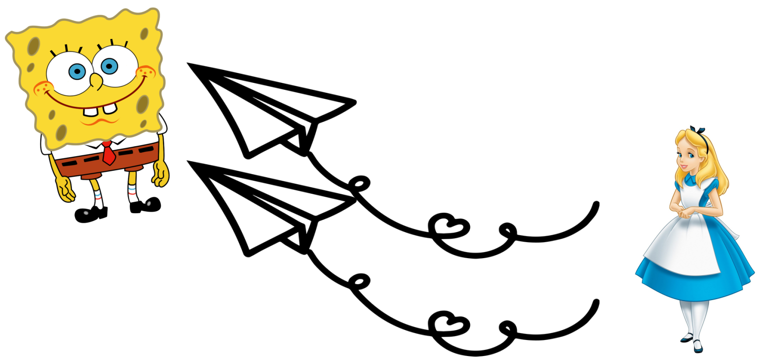

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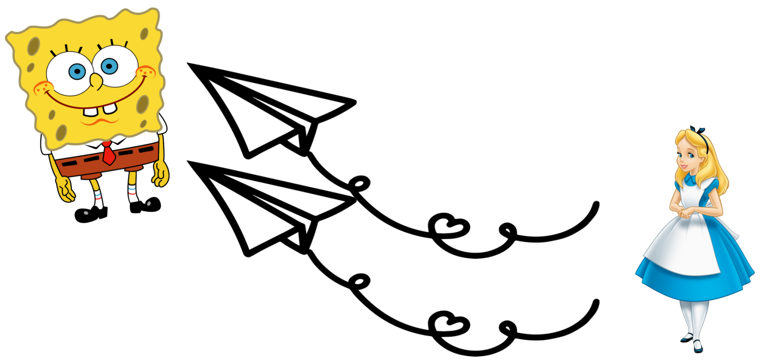

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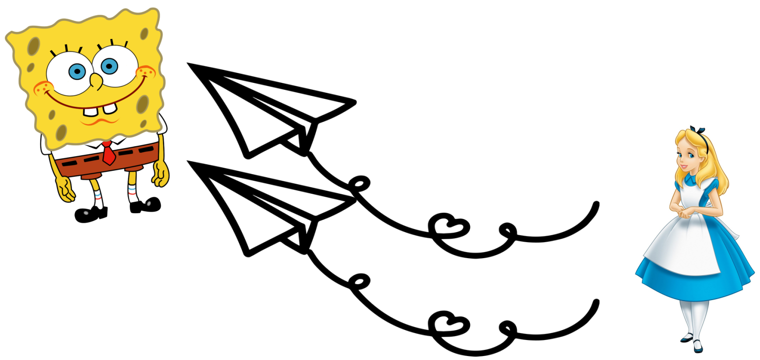

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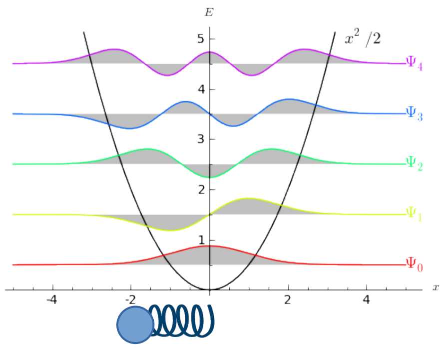

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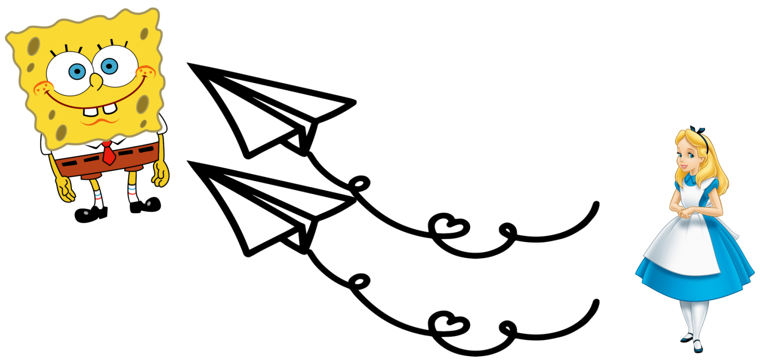

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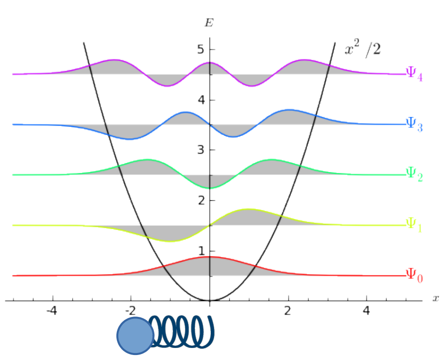

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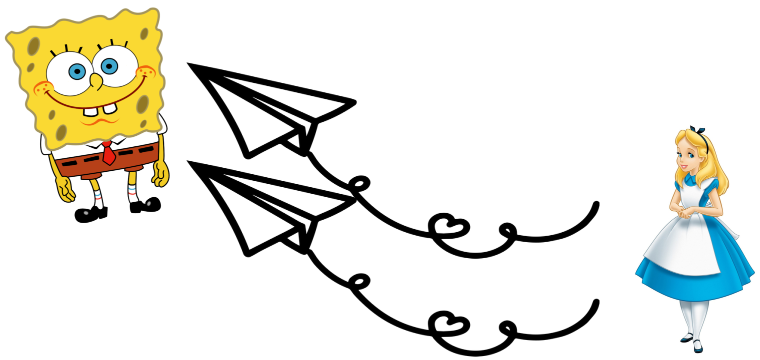
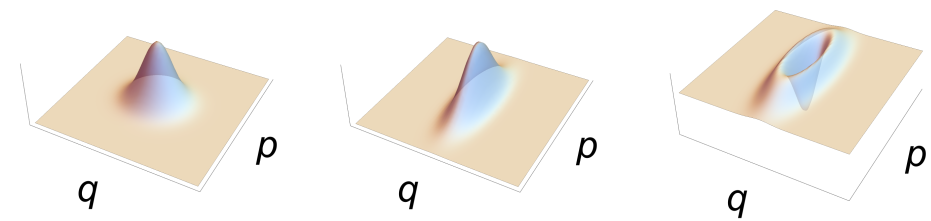


Phase space:

$$W_\rho(\boldsymbol{q},\boldsymbol{p}) = (2\pi)^{-2n} \int \mathrm{d}^n \boldsymbol{y} \left\langle \boldsymbol{q} - \frac{\boldsymbol{y}}{2} \middle| \hat{\rho} \middle| \boldsymbol{q} + \frac{\boldsymbol{y}}{2} \right\rangle e^{i\boldsymbol{p}\cdot\boldsymbol{y}}$$



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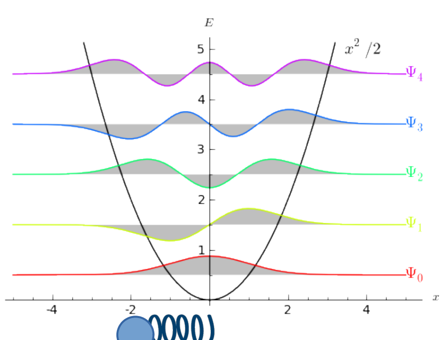

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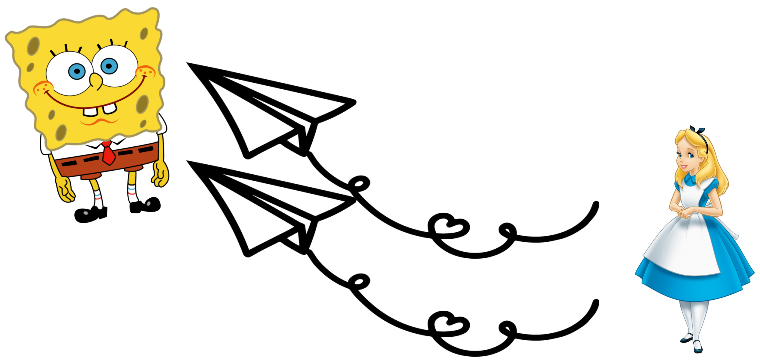

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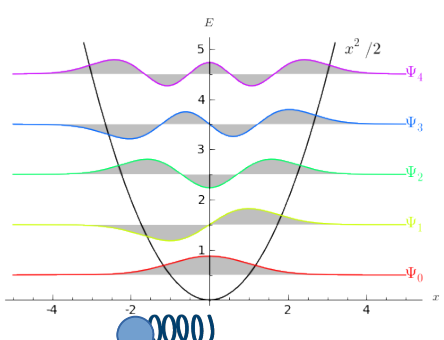

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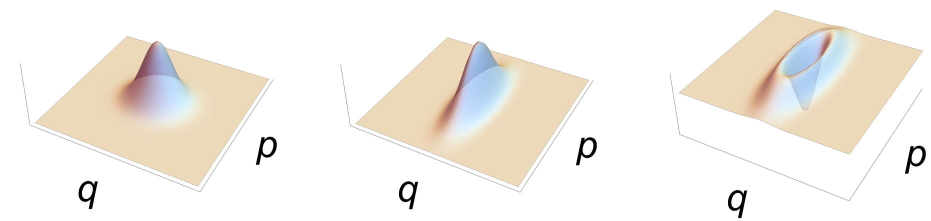

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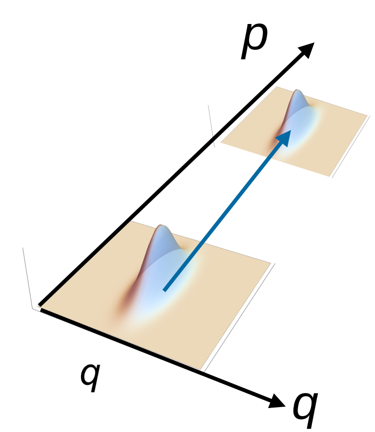

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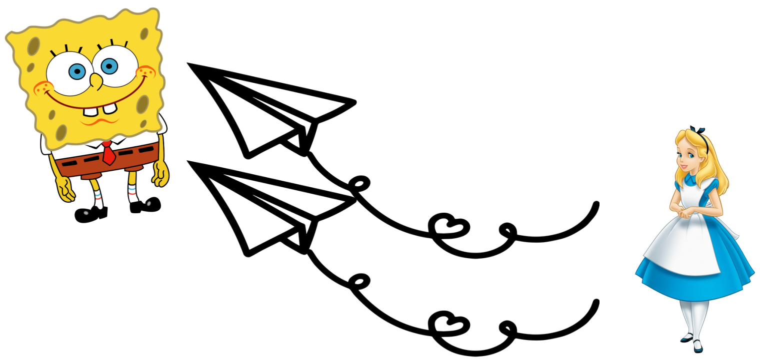

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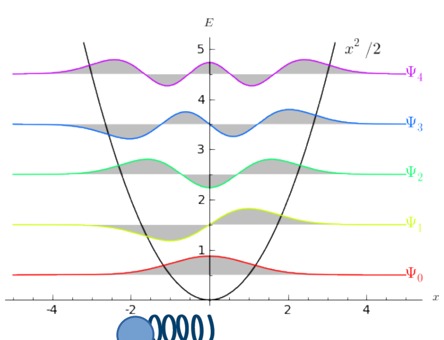

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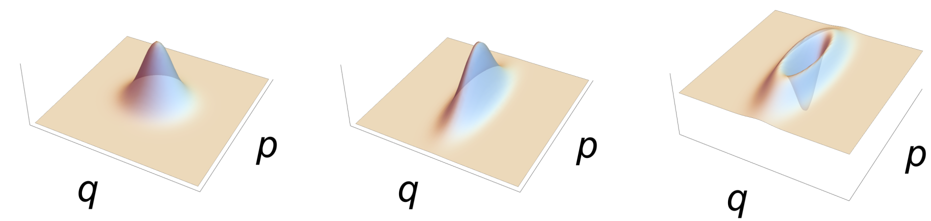

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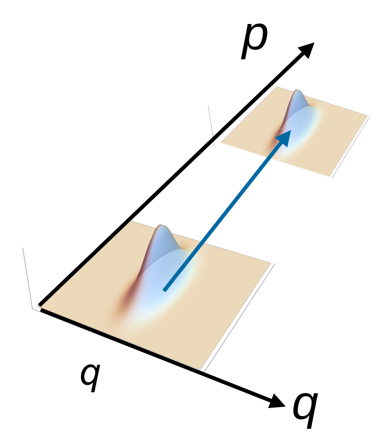

$$D(\boldsymbol{\xi})D(\boldsymbol{\eta}) = e^{-i2\pi\boldsymbol{\xi}^T J\boldsymbol{\eta}} D(\boldsymbol{\eta})D(\boldsymbol{\xi})$$

Encoding qubits on a lattice

“Logical subspace”: finite photon number, odd/even photon number, or...**translation symmetries**!

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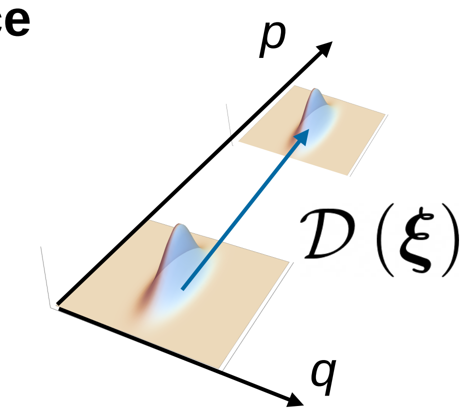
Gottesman, Kitaev, Preskill PRA 64 (2001)



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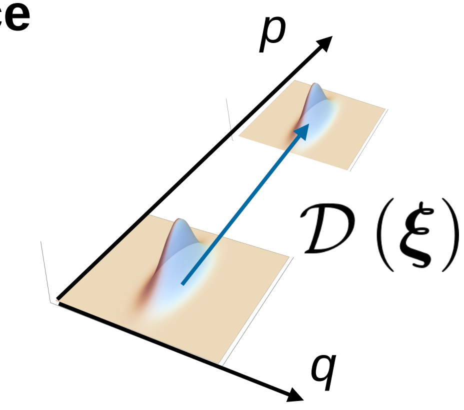

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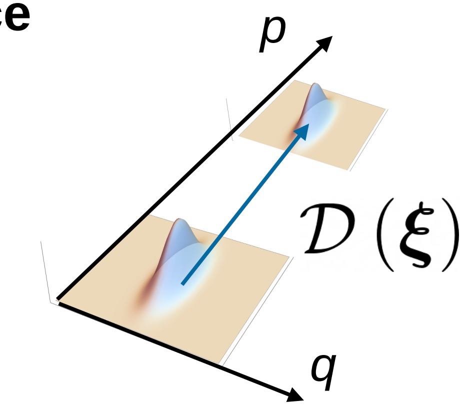

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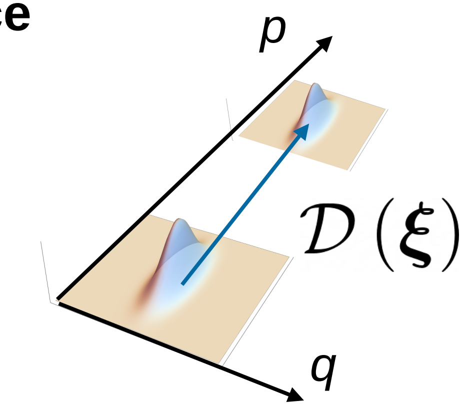

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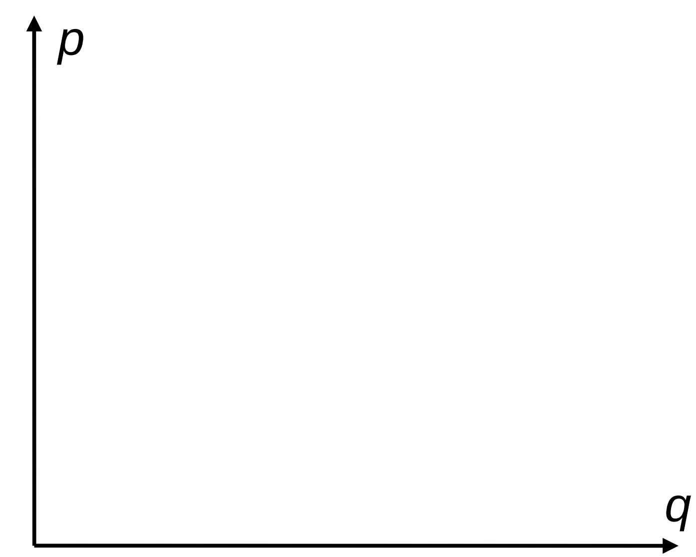

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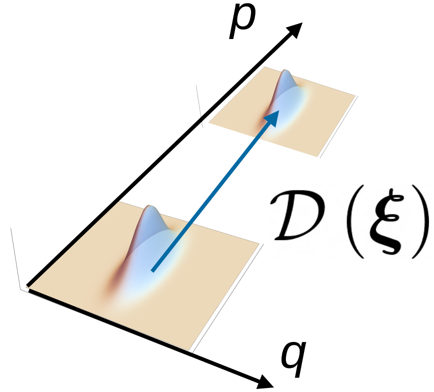

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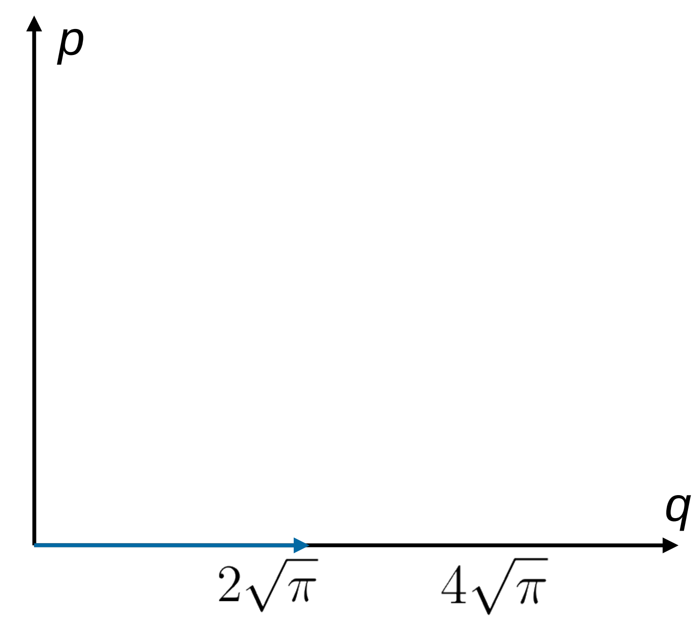

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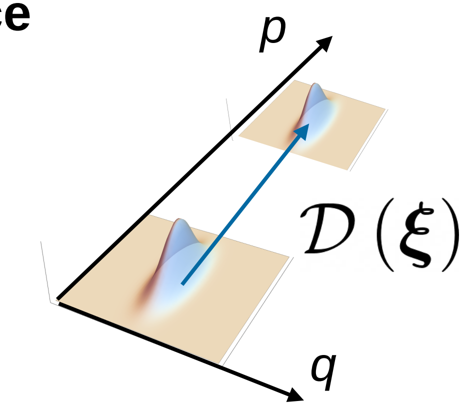

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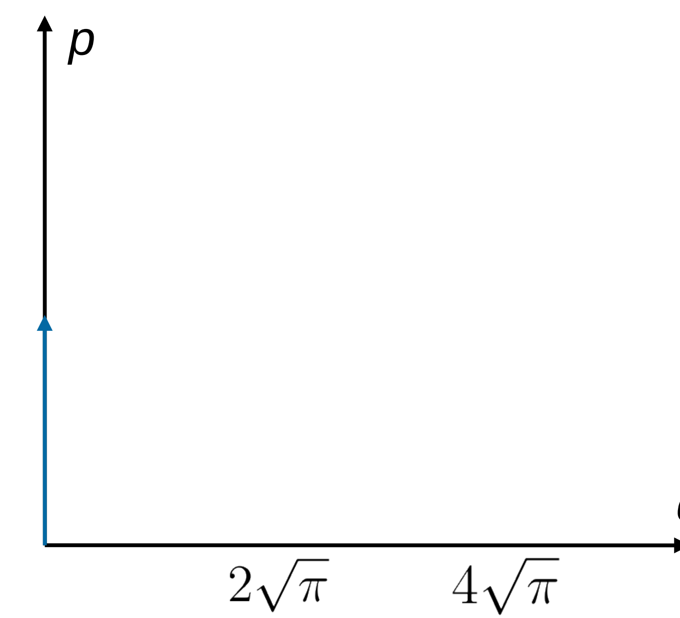

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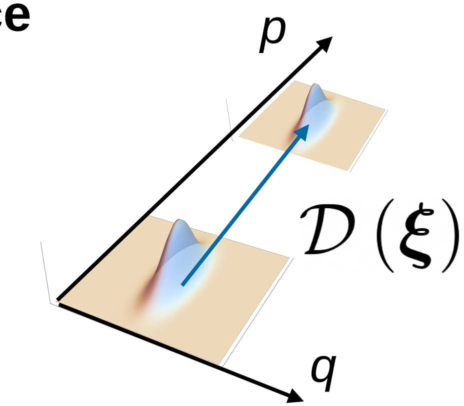

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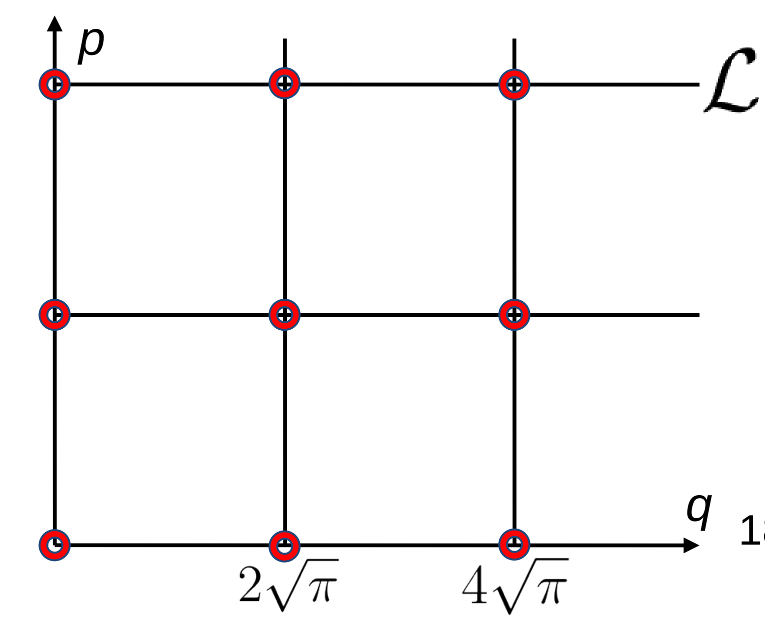

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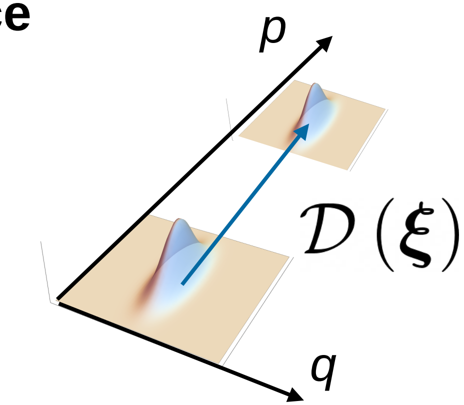

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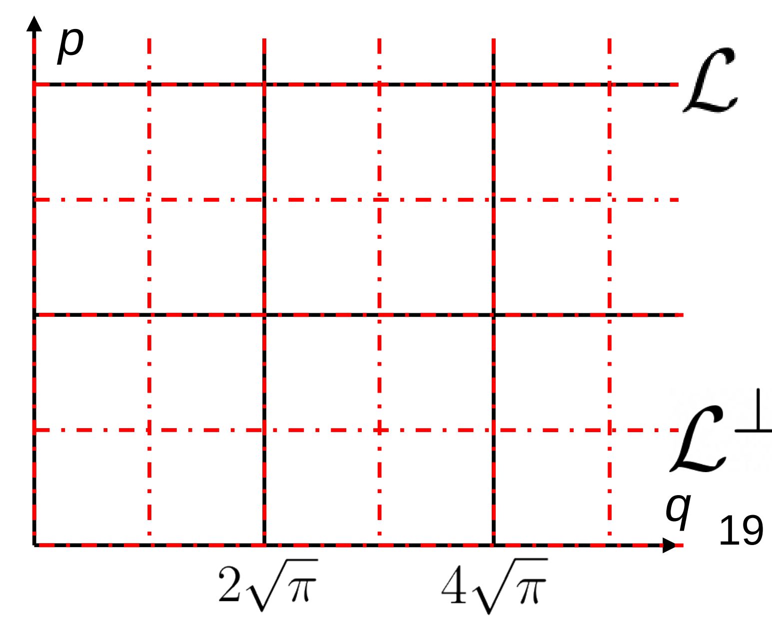

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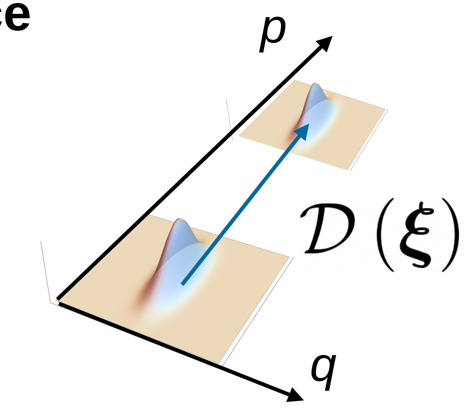

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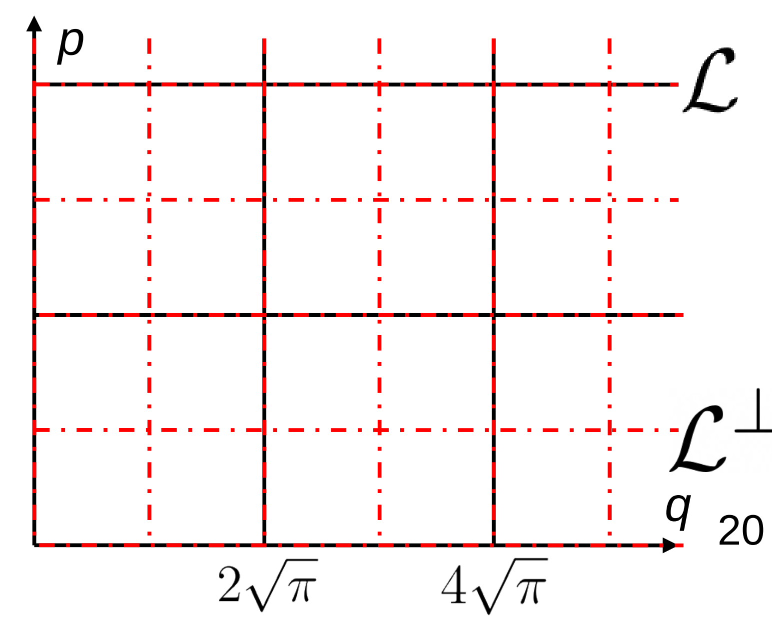

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Albert et al, PRA 97 (2018)

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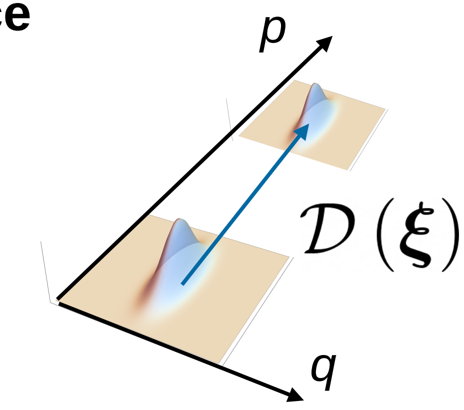

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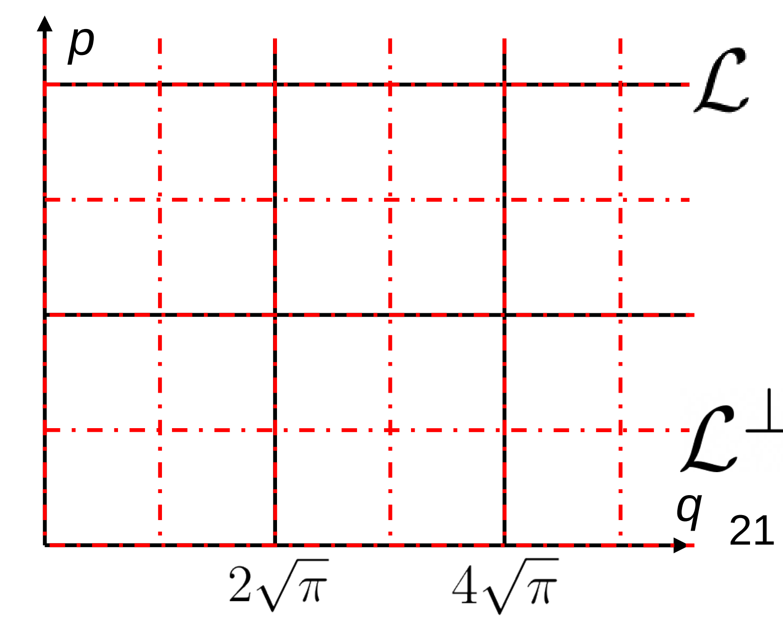

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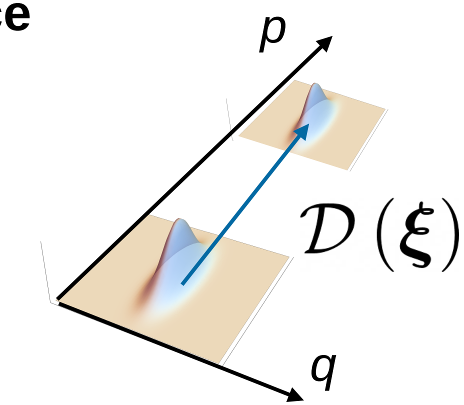

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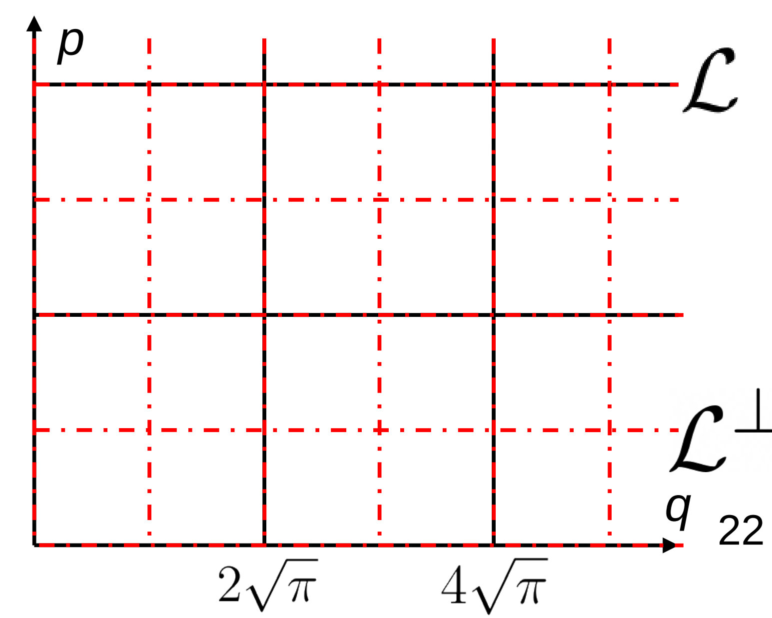

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- Can be effective qubits, combined with qubit codes

Vuillot et al, PRA 99 (2019) *Noh&Chamberland PRA 101 (2020)* *Bourassa et al, Quantum 5 (2021)*

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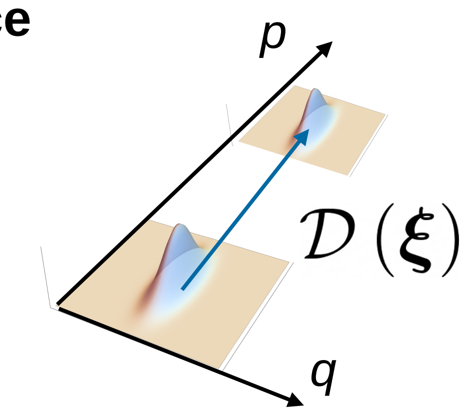

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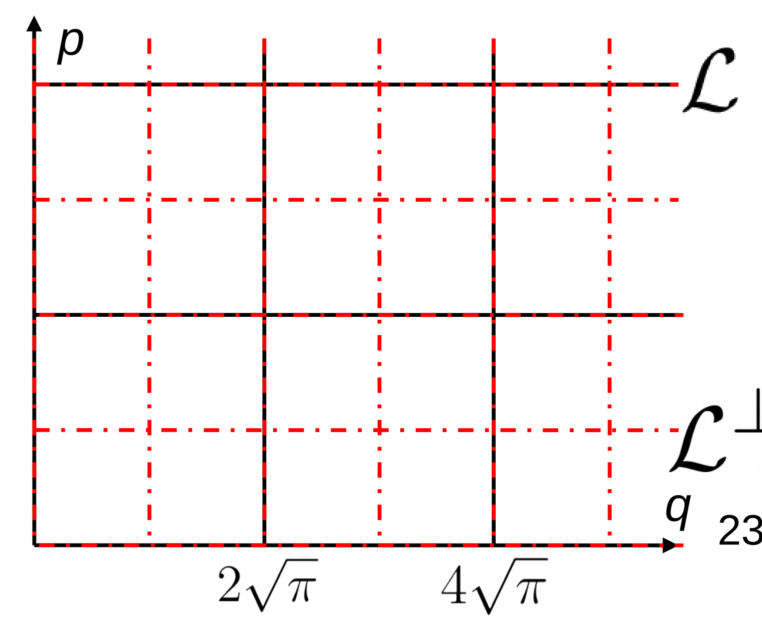

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  Albert et al, PRA 97 (2018)
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Encoding qubits on a lattice

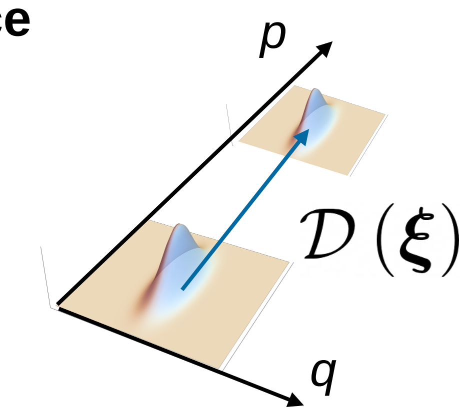
"Logical subspace": finite photon number, odd/even photon number, or...**translation symmetries**!

Grid codes: **stabilized** by (commuting) **displacement** operators → underlying **lattice**

Gottesman, Kitaev, Preskill PRA 64 (2001)

$$\mathcal{S} = \langle D(\boldsymbol{\xi}_1), \ldots, D(\boldsymbol{\xi}_{2n}) \rangle \Rightarrow \text{Code: } D(\boldsymbol{\xi}_j)\,|\psi\rangle = |\psi\rangle$$

$$\boldsymbol{\xi}_j^T J \boldsymbol{\xi}_k \in \mathbb{Z}$$



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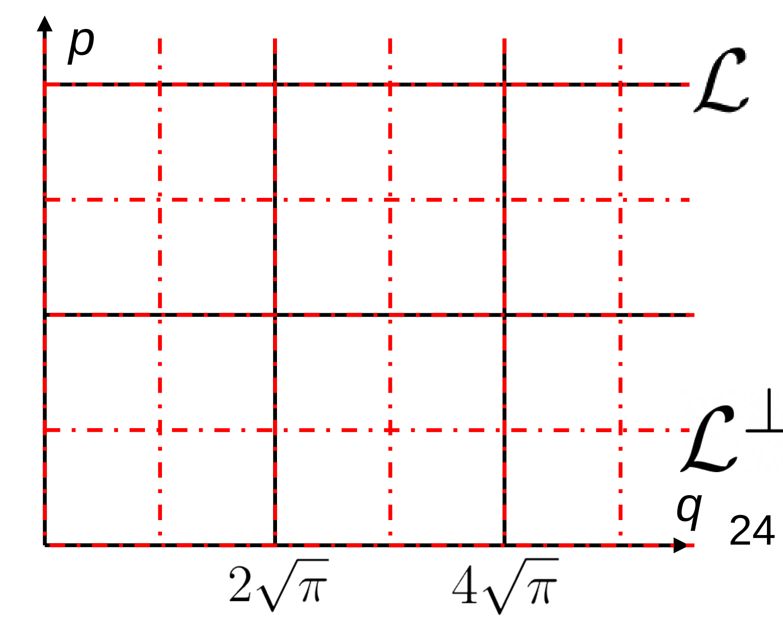

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  Flühmann et al, Nature 566 (2019) *Campagne-Ibarcq et al, Nature 584 (2020)*

The lattice point of view

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For exponential noise suppression: more oscillators

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Outline

1. Lattice formalism
2. Code properties from lattice bases
3. Symplectic operations
4. Distance bounds for GKP codes
5. Decoding problem and Θ functions
6. GKP codes beyond concatenation

Lattice formalism

$$\mathcal{S} = \left\langle D\left(\boldsymbol{\xi}_1\right), \ldots, D\left(\boldsymbol{\xi}_{2n}\right)\right\rangle$$

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Gottesman, Kitaev, Preskill PRA 64 (2001)

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Examples

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Exploit basis manipulations/properties to study codes

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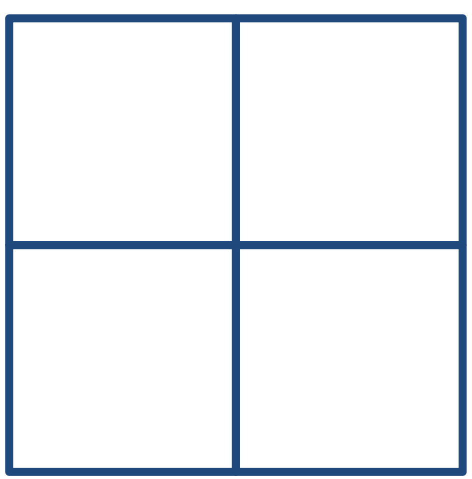

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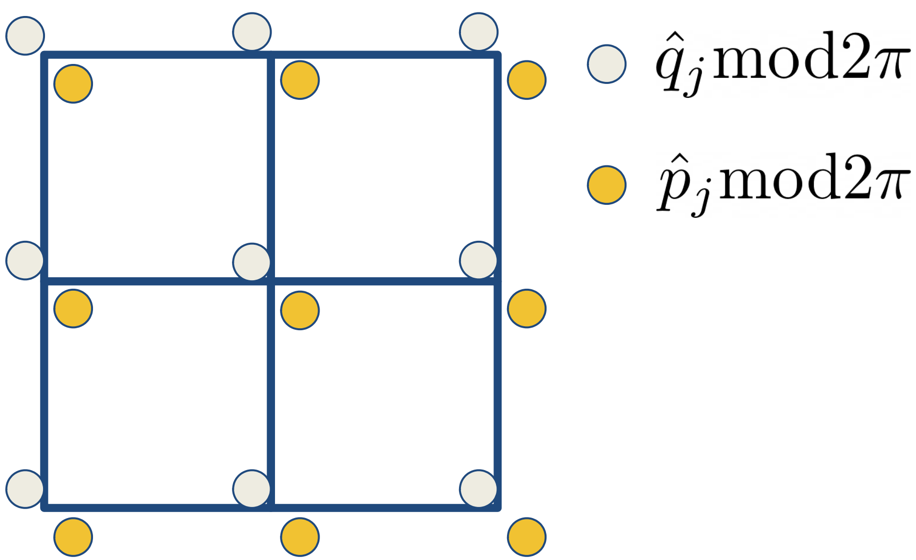

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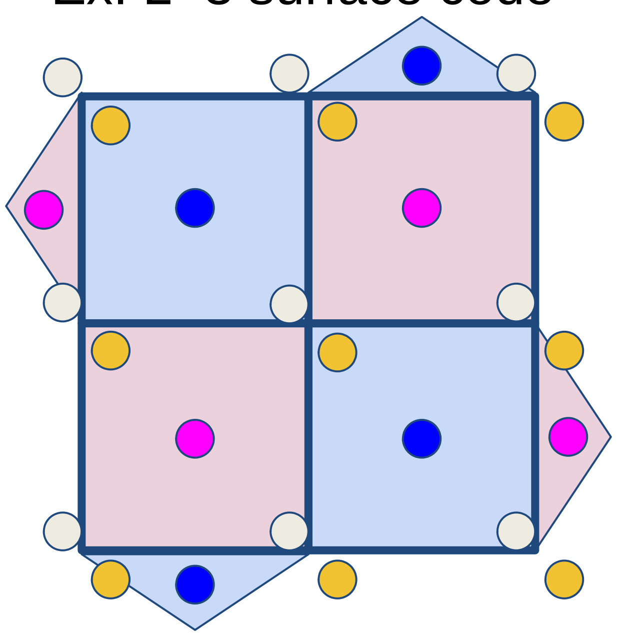

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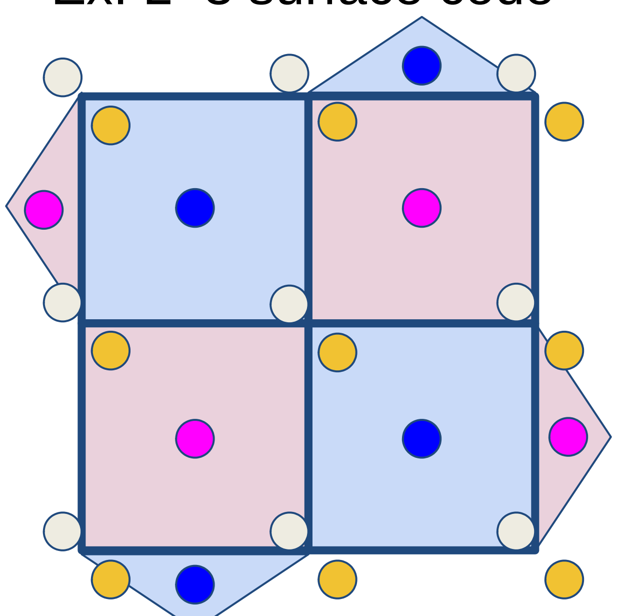

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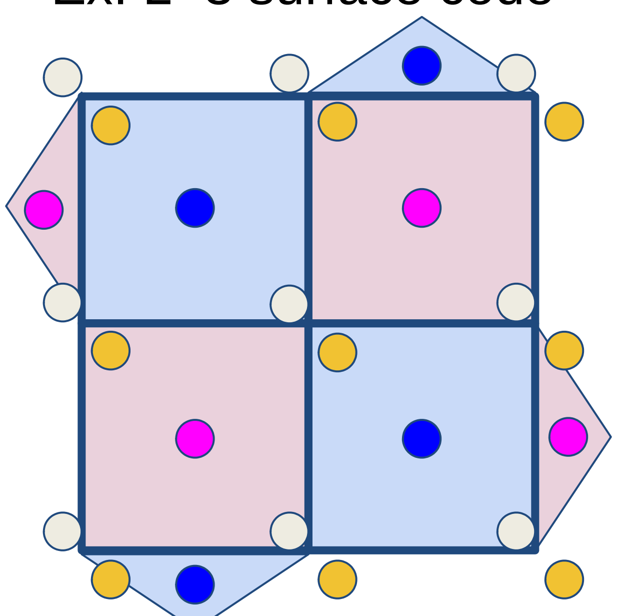

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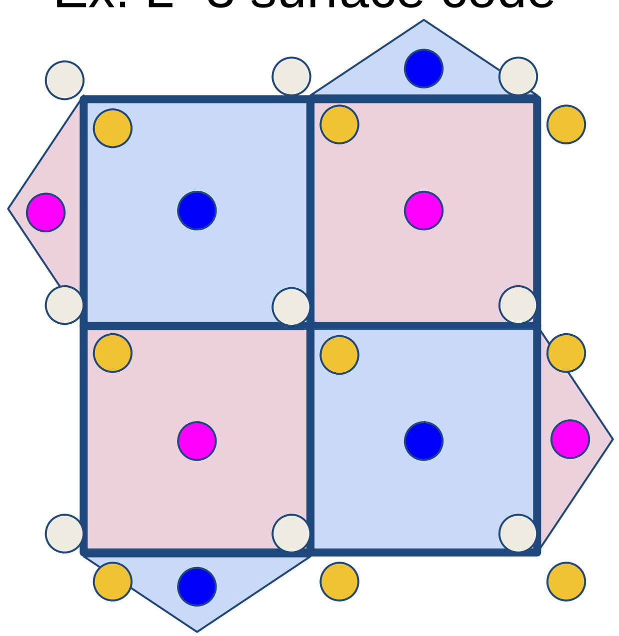

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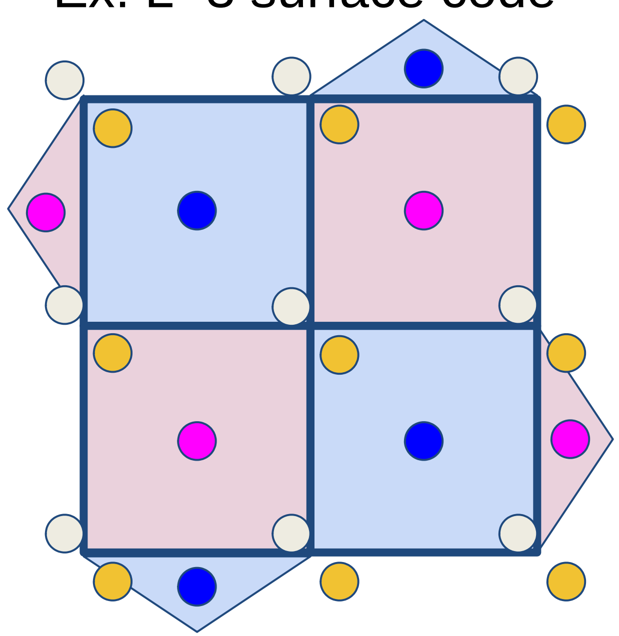

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$\Downarrow$ lattice basis reduction

$$M_{\min}$$

2n = ***18*** stab. gens.

can do respecting weights, geometric locality!

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Multi-mode generalization of *Hänggli, Heinze, König, PRA 102 (2020)*

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Symplectic operations

$$U_S = \exp\left(-i\hat{\boldsymbol{x}}^T H \hat{\boldsymbol{x}}\right), \quad \mathcal{S} \mapsto U_S^\dagger \mathcal{S} U_S \Leftrightarrow M \mapsto M S^T, \quad M^\perp \mapsto M^\perp S^T$$

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Theorem (symplectically equivalent codes):

$$\boxed{\text{Given } \mathcal{L}(M), \mathcal{L}(N),\ \exists S \mid M = NS^T \text{ iff } MJM^T = NJN^T \text{ (in canonical form)}}$$

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Generalizes to higher logical dimensions $M_\square J M_\square^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathrm{diag}\{d_1, \ldots, d_n\}$

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Measure of "squeezing"

Conclusions

Summary

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- Lattice bases: link to experimental hardness, resource savings
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Thank you!

Paper coming soon!