Gottesman-Kitaev-Preskill bosonic error correcting codes: a lattice perspective

[arXiv:2109.14645](https://arxiv.org/abs/2109.14645)

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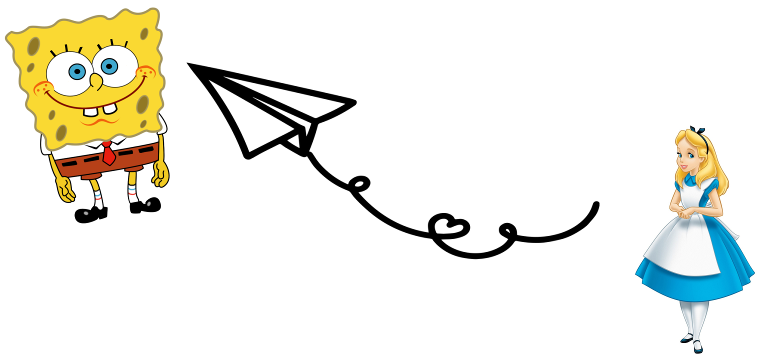
Alexander von Humboldt Stiftung/Foundation

Freie Universität Berlin

(Quantum) Error correction and harmonic oscillators



Information always encoded in phys. syst.

$\Rightarrow$ Always subject to **noise**

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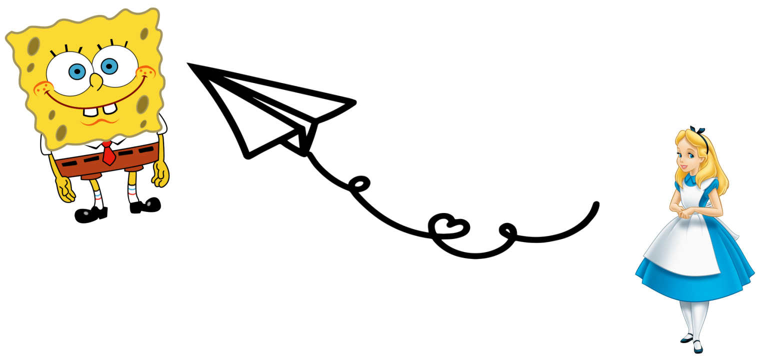

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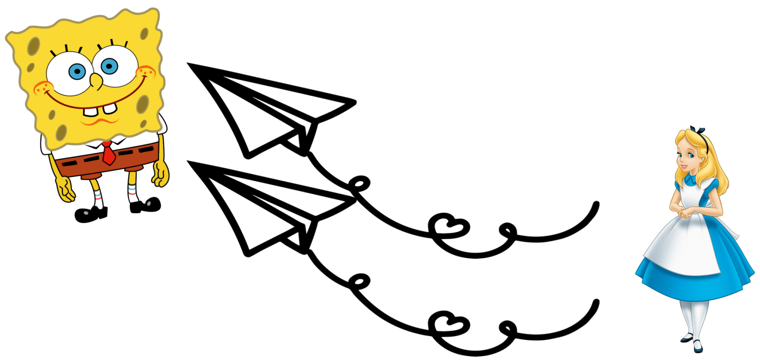

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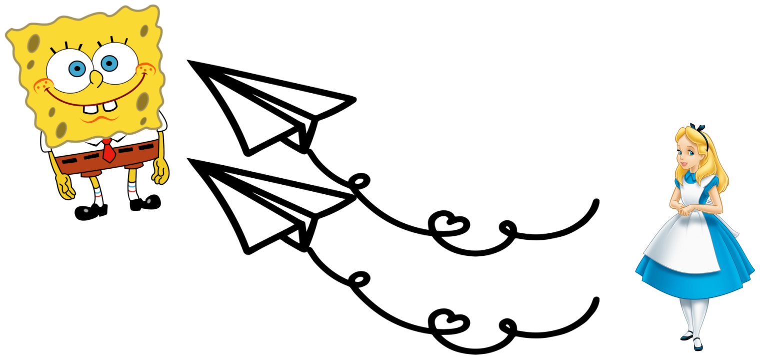

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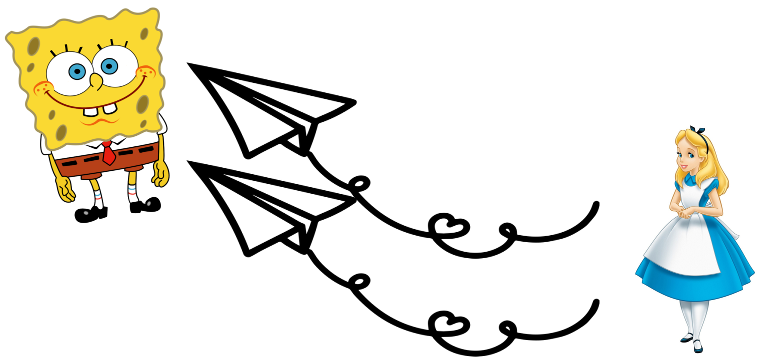

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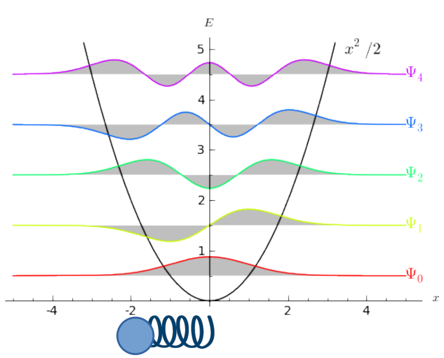

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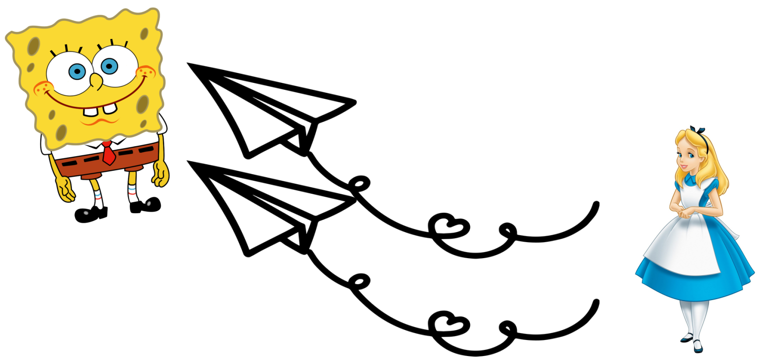

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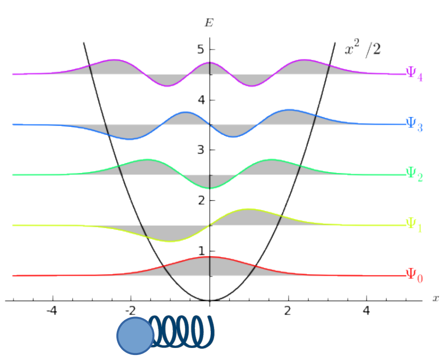

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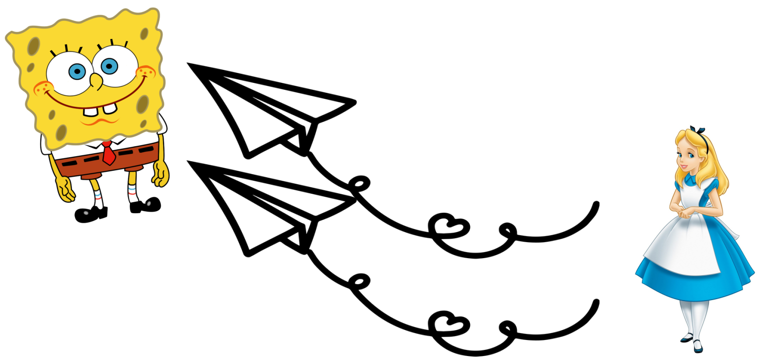

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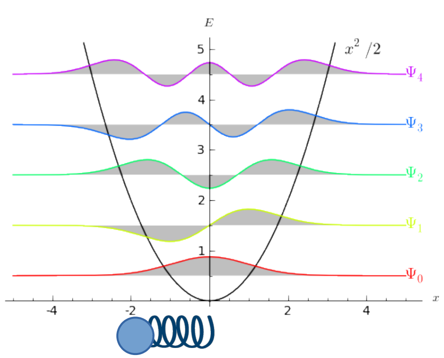

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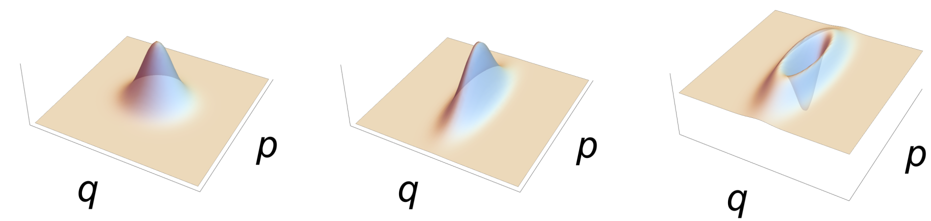

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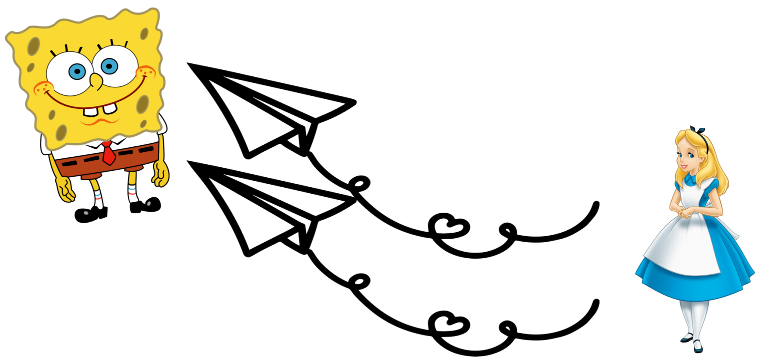

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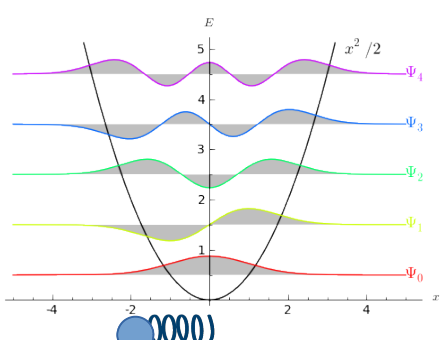

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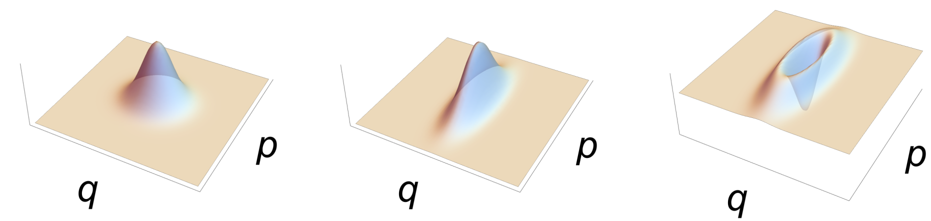

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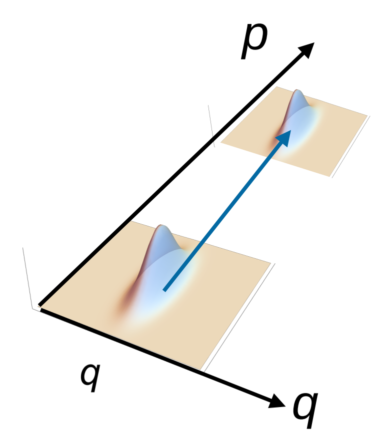

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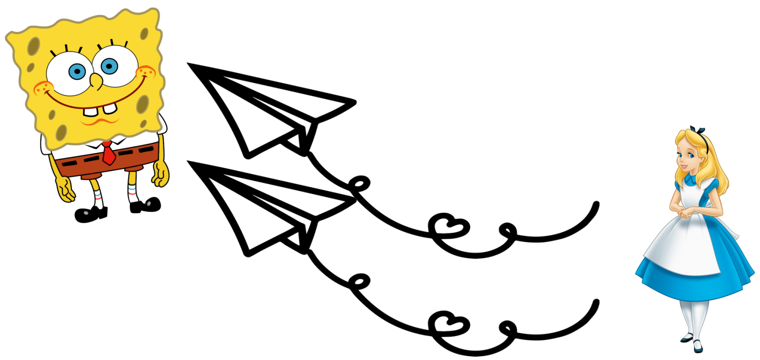

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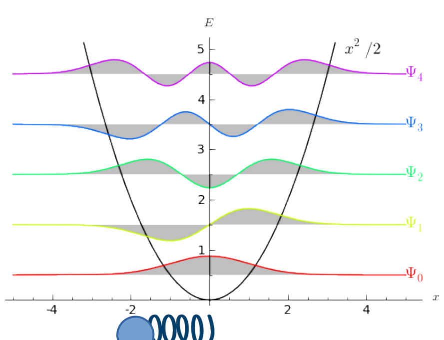

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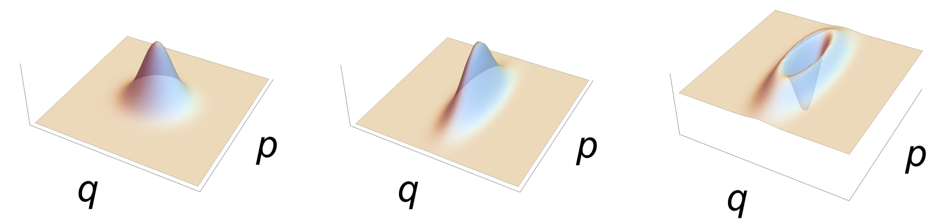

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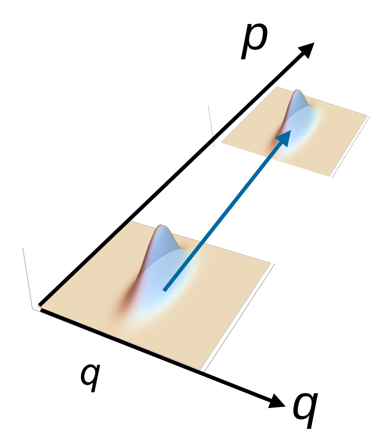

$$D(\boldsymbol{\xi})D(\boldsymbol{\eta}) = e^{-i2\pi\boldsymbol{\xi}^T J\boldsymbol{\eta}} D(\boldsymbol{\eta})D(\boldsymbol{\xi})$$

Encoding qubits on a lattice

“Logical subspace”: finite photon number, odd/even photon number, or...**translation symmetries**!

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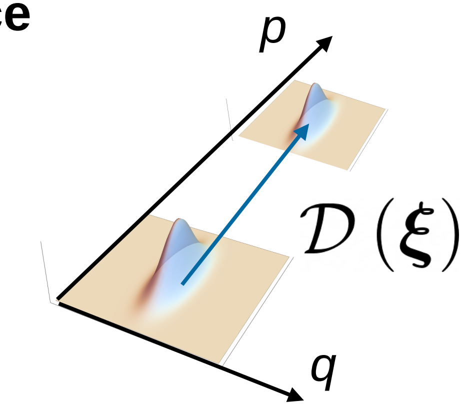

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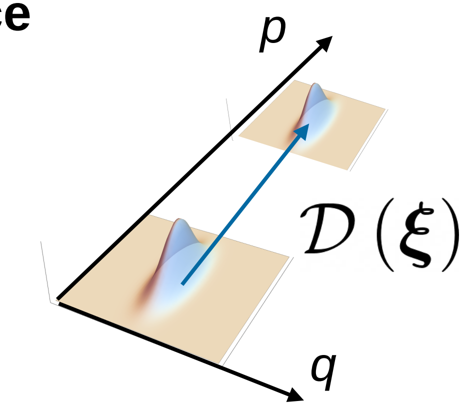

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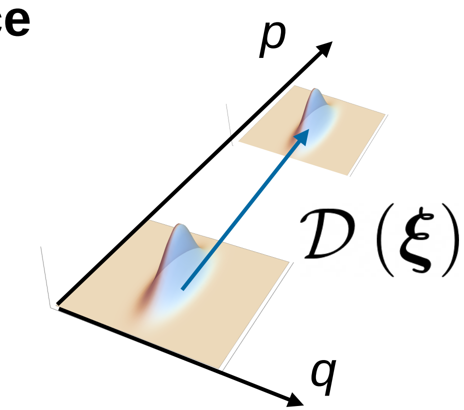

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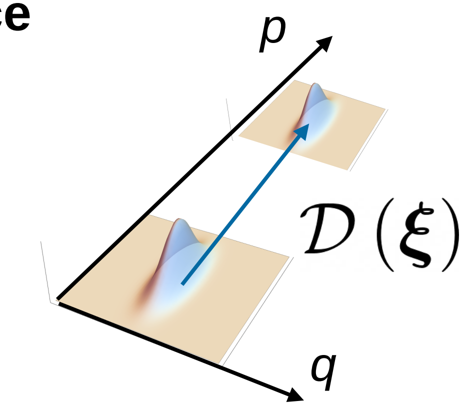

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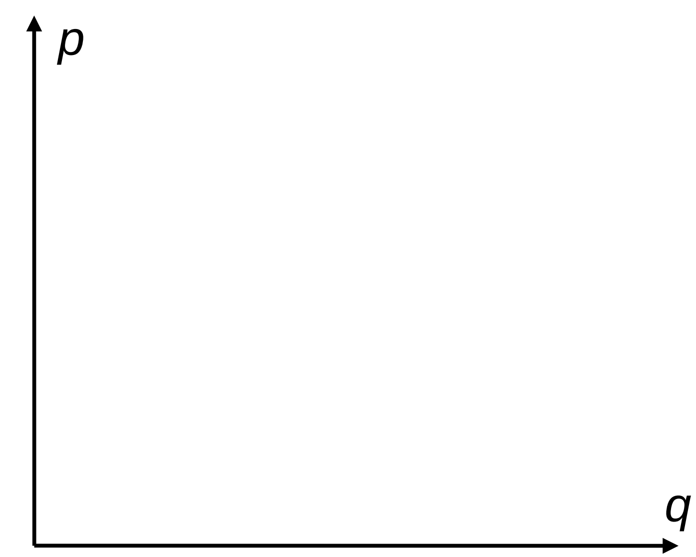

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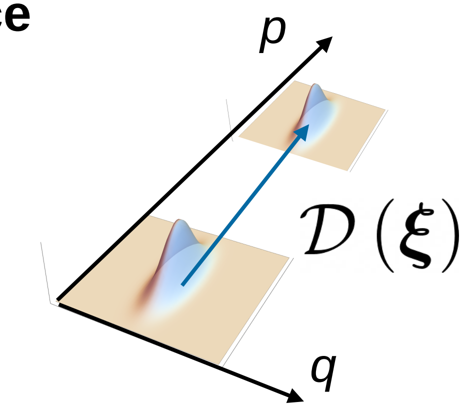

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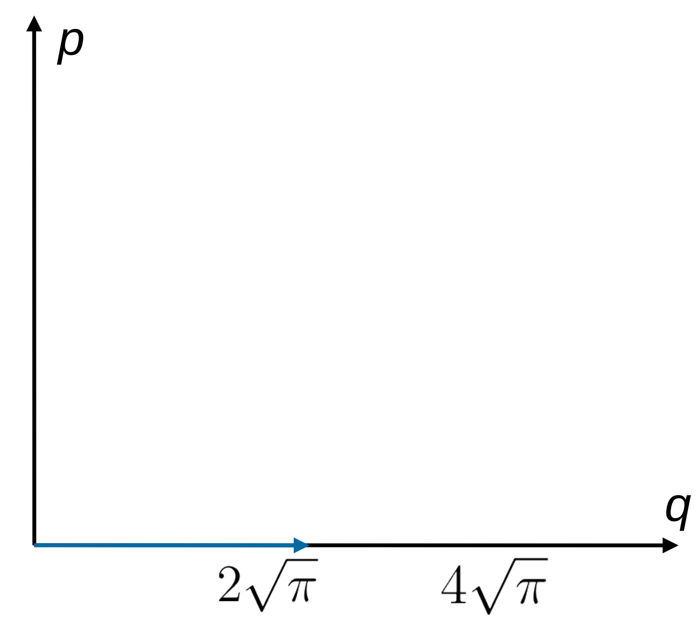

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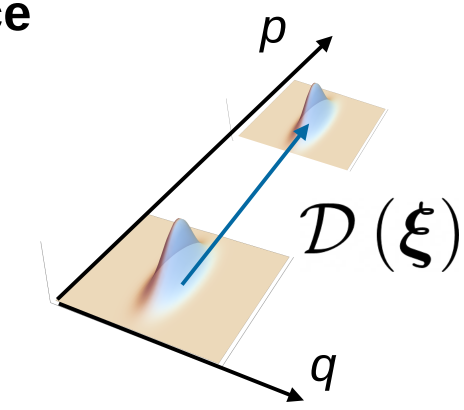

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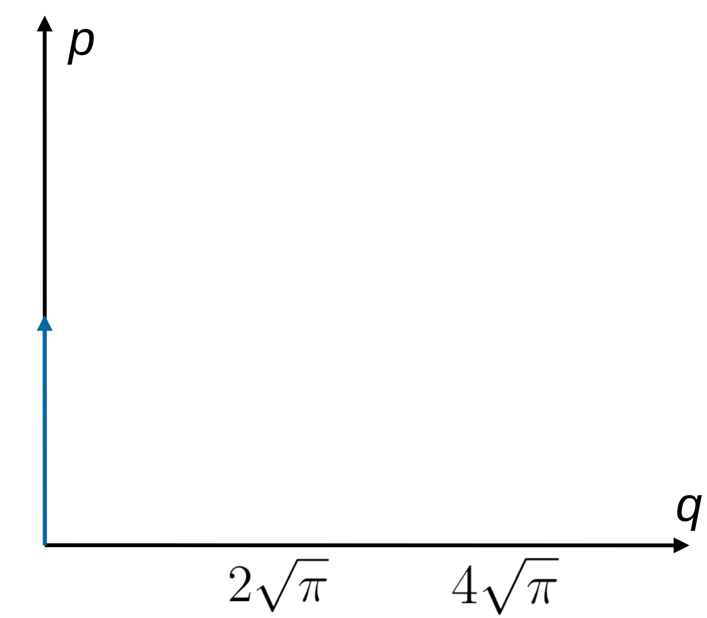

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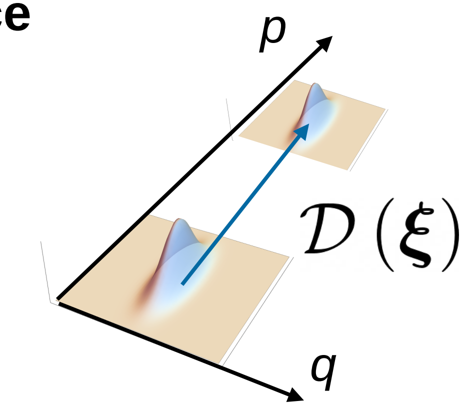

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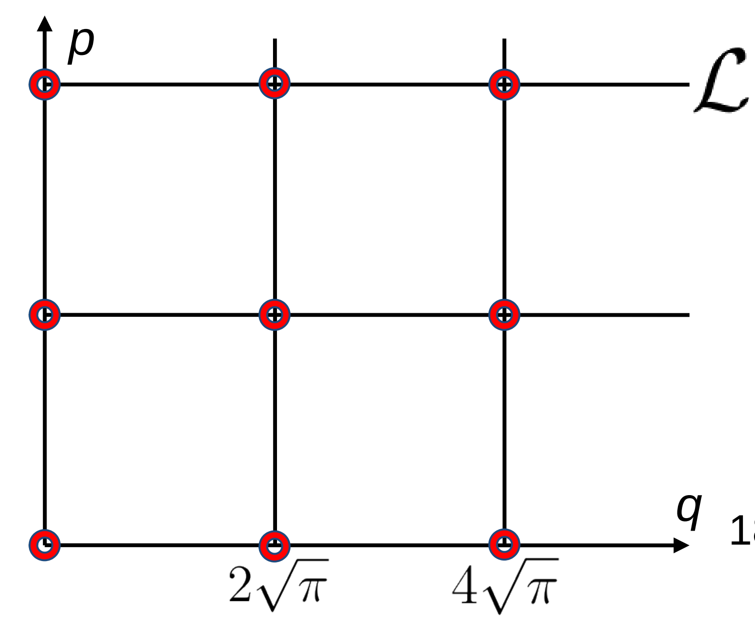

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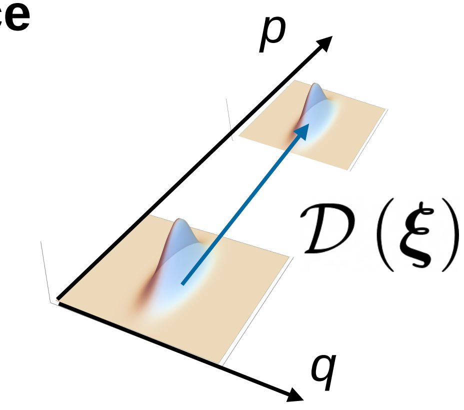

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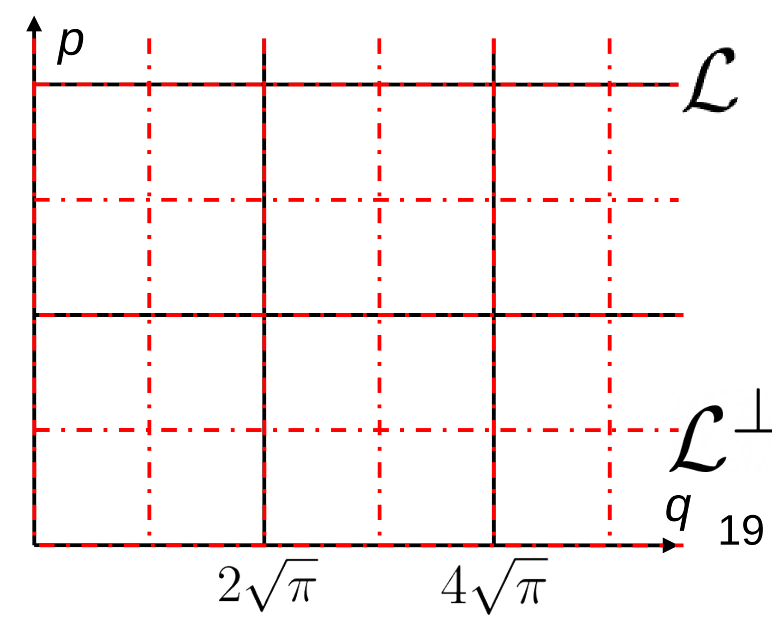

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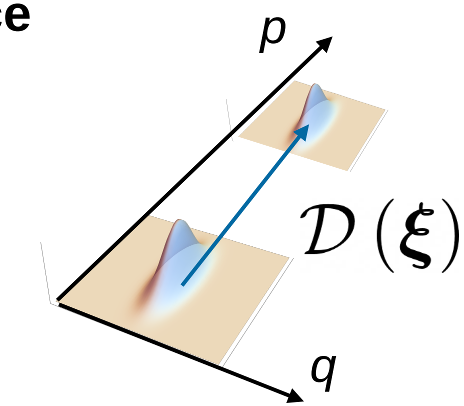

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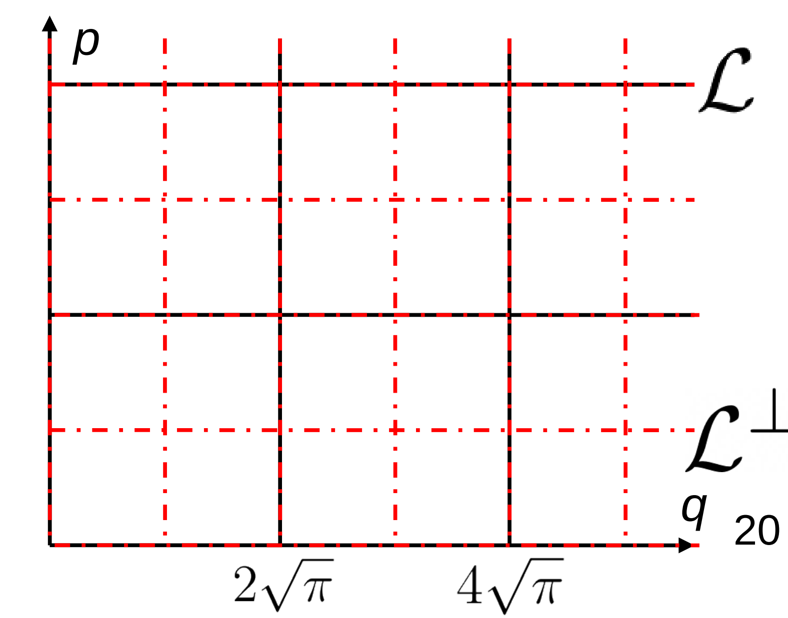

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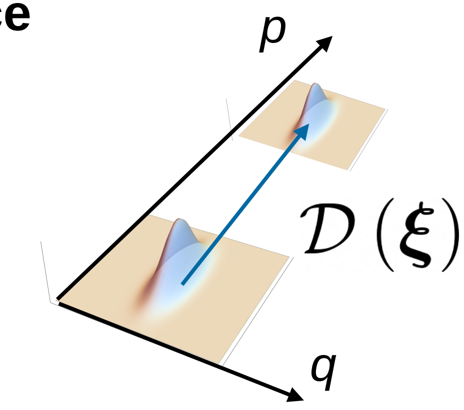

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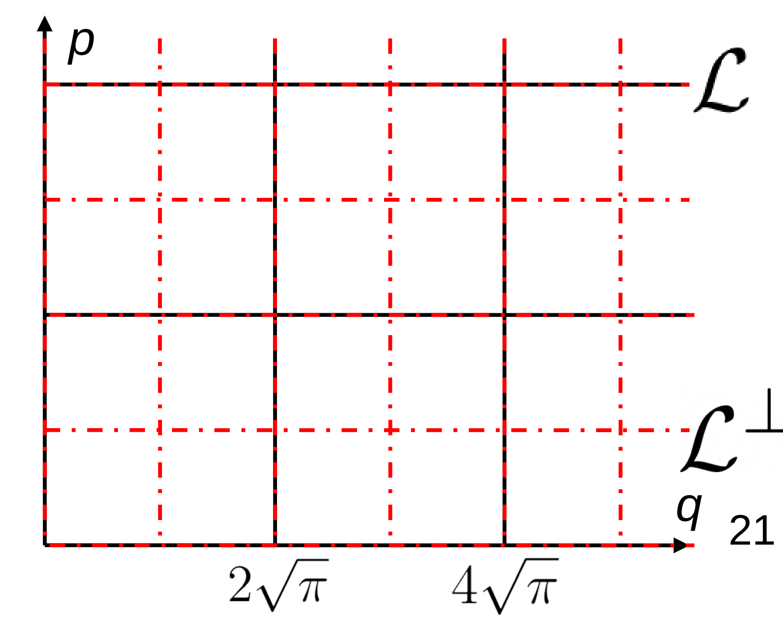

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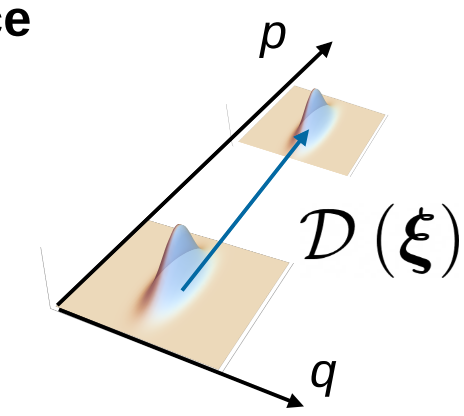

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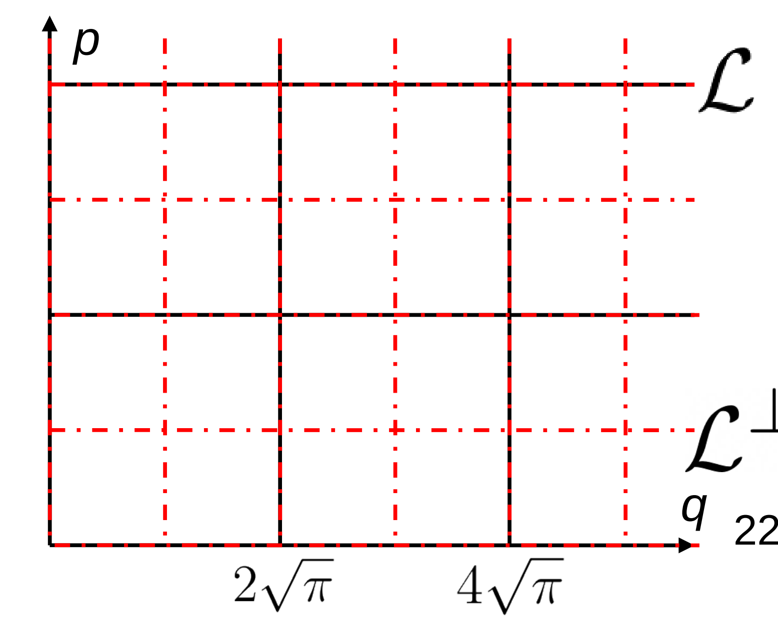

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- Can be effective qubits, combined with qubit codes

Vuillot et al, PRA 99 (2019) *Noh&Chamberland PRA 101 (2020)* *Bourassa et al, Quantum 5 (2021)*

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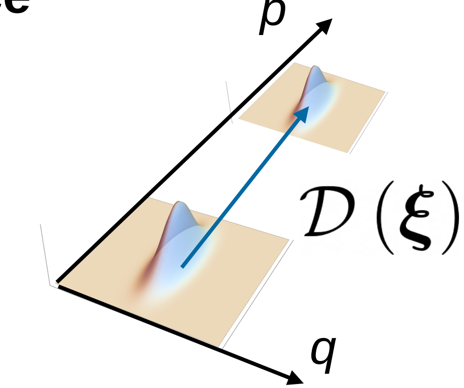

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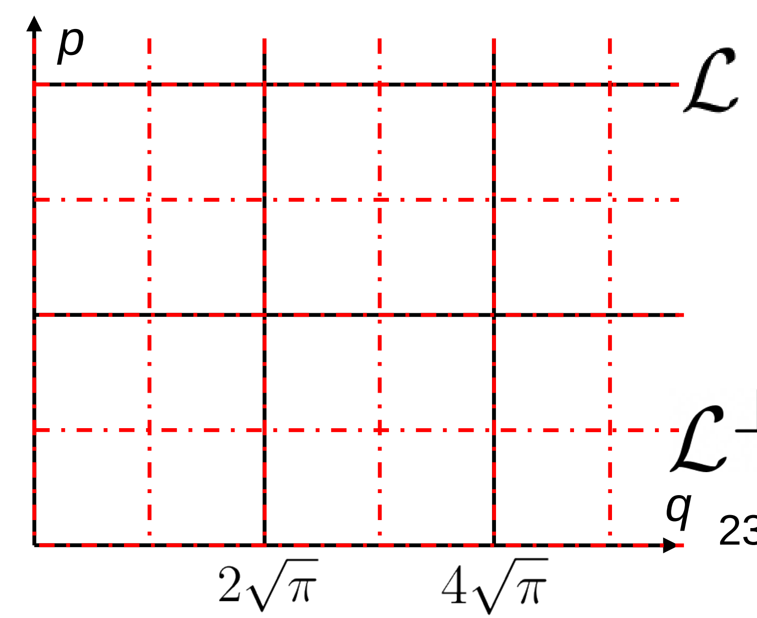

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- Logical Clifford = Gaussian operations
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Vuillot et al, PRA 99 (2019) *Noh&Chamberland PRA 101 (2020)* *Bourassa et al, Quantum 5 (2021)*

- Can protect CV systems
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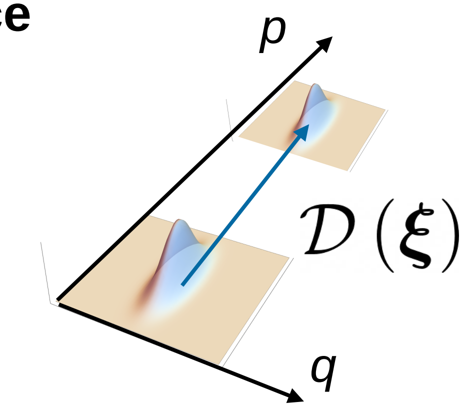
Encoding qubits on a lattice

"Logical subspace": finite photon number, odd/even photon number, or...**translation symmetries**!

Grid codes: **stabilized** by (commuting) **displacement** operators → underlying **lattice**

Gottesman, Kitaev, Preskill PRA 64 (2001)



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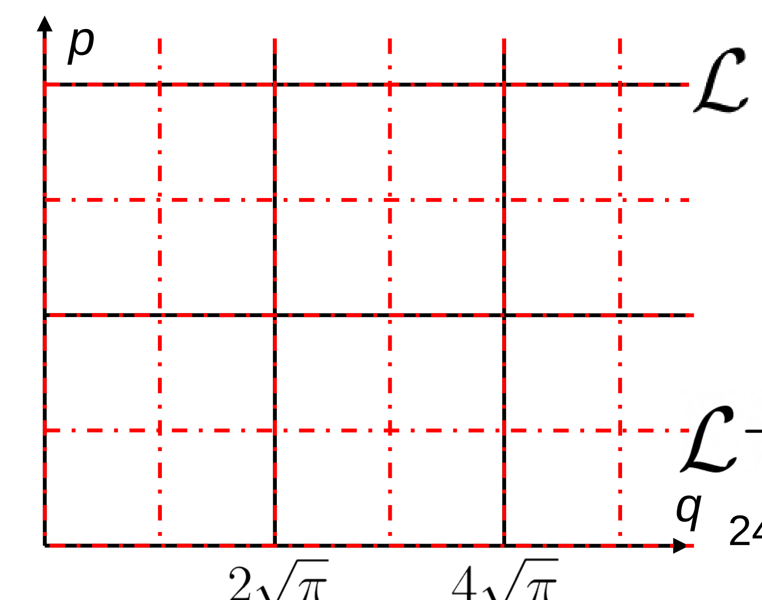

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  Flühmann et al, Nature 566 (2019) *Campagne-Ibarcq et al, Nature 584 (2020)*

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For exponential noise suppression: more oscillators

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Outline

1. Lattice formalism
2. Code properties from lattice bases
3. Symplectic operations
4. Distance bounds for GKP codes
5. Decoding problem and Θ functions
6. GKP codes beyond concatenation

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Results

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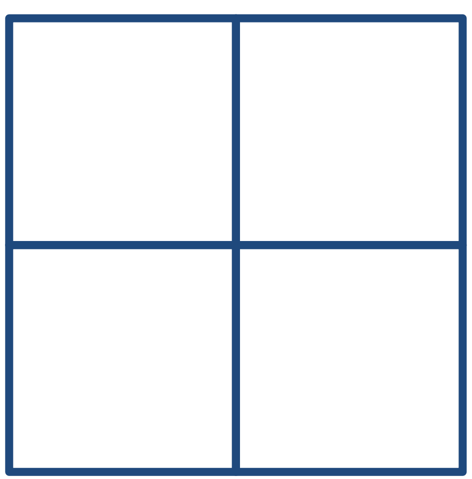

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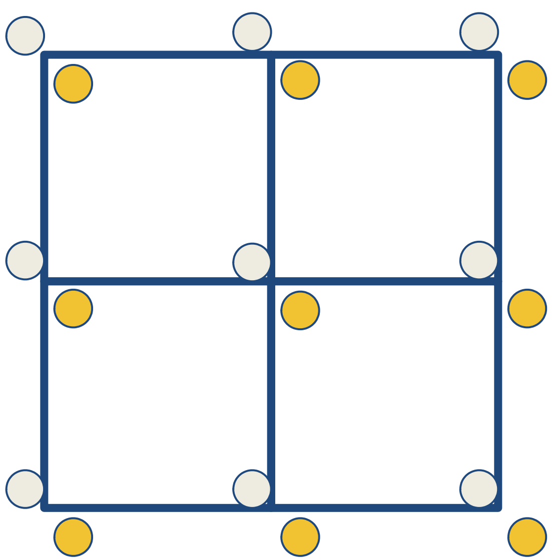

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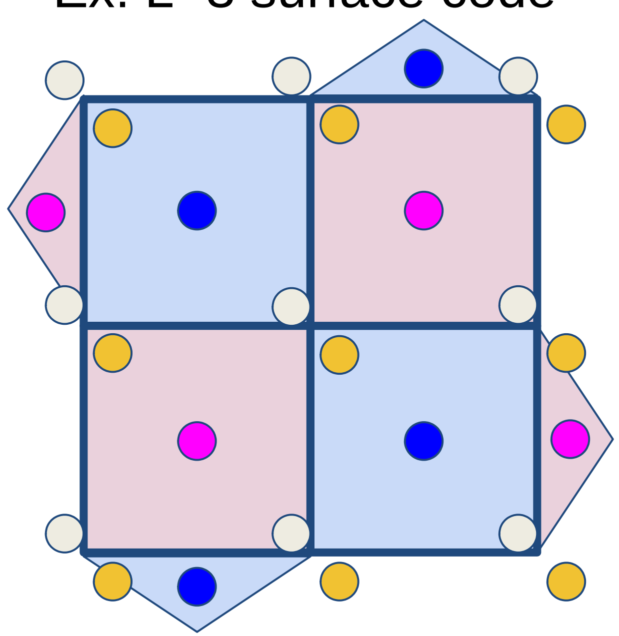

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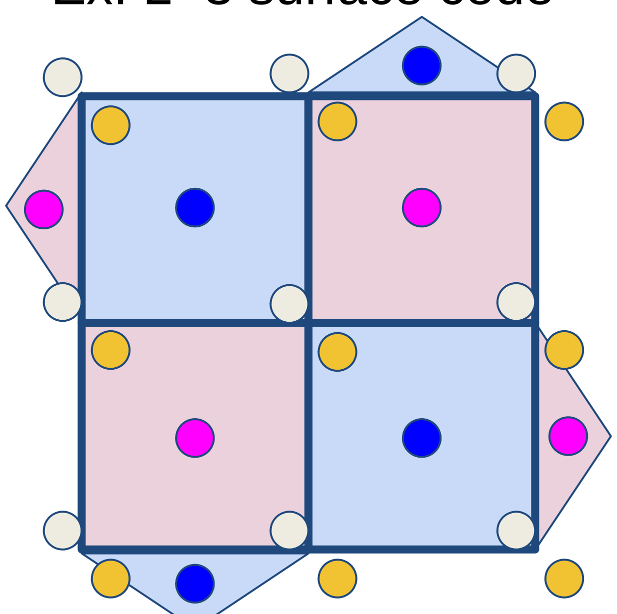

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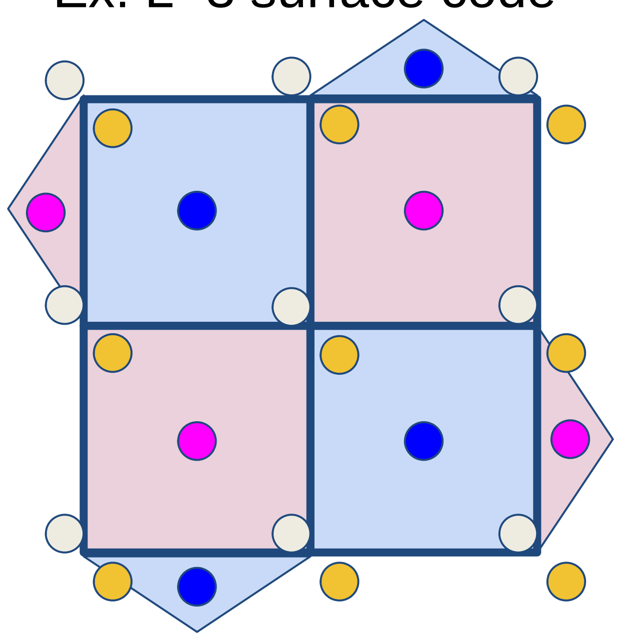

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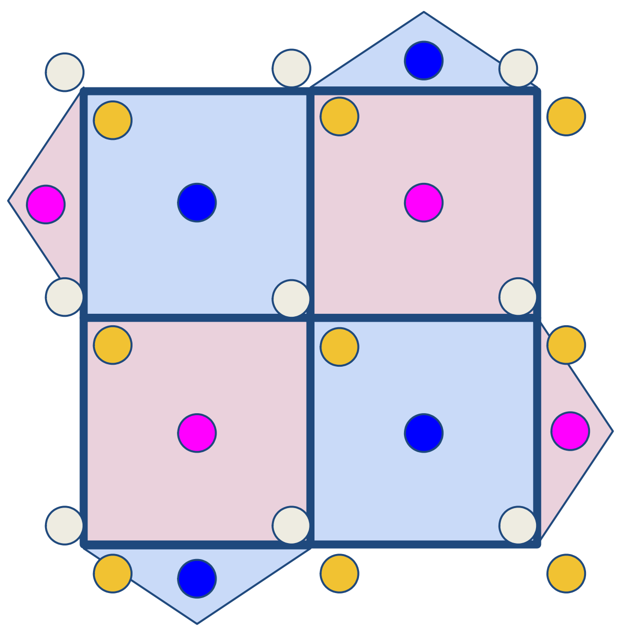

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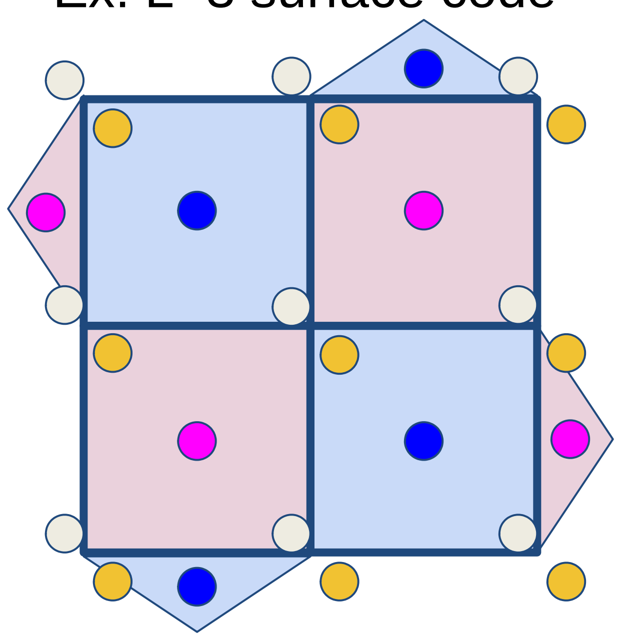

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$$M_{\min}$$

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can do respecting weights, geometric locality!

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then scale back.

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$$\boxed{\text{Any code with } d = 2 \text{ is s.e. to } \mathcal{S}_\square^{(2)} = \left\langle e^{i2\sqrt{\pi}\hat{q}_1}, e^{-i2\sqrt{\pi}\hat{p}_1}, e^{i\sqrt{\pi}\hat{q}_j}, e^{i\sqrt{\pi}\hat{p}_j} \right\rangle,\ j > 1}$$

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Example of **inequivalent codes**: 2 qubits in 2 modes can have $D = \mathrm{diag}(4,1)$ or $D = \mathrm{diag}(2,2)$

Distance of GKP codes

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with $\lambda_j(\mathcal{L}) = \min_{\mathbb{R}^+}\{r \mid \mathcal{L} \text{ contains } j \text{ l.i. vectors with } ||\boldsymbol{x}|| \leq r\}$ (successive minima)

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Measure of "squeezing"

Distance of GKP codes: Θ functions

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Lattice Θ function:

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$$q = e^{i\pi z}$$

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Theorem:

$\boxed{\text{Distance of GKP code } \mathcal{L}(\mathrm{M}) \text{ completely specified by distance distribution } (\mathcal{D}, \mathrm{N}_\delta)}$

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The distance of a GKP code can be estimated through logarithmic fit for small *q*

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$$D\left(\boldsymbol{e}\right)\left|\psi\right\rangle$$

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upshot:
Θ functions are ubiquitous in lattice theory, estimate those for approx MLD!

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New code construction: glued lattices

Inspired by: *T. Gannon,* Lattices and theta functions, *PhD thesis (1991)* (applied to string theories)

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Conjecture: distance computed similarly to concatenated codes (but strongly depends on *G*)

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Symplectically integer with $G_2 = M_2 M_2^T$ integer

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Conclusions

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Thank you! arXiv:2109.14645