Can effective descriptions of bosonic systems be considered complete?

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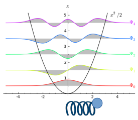
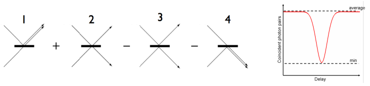
Introduction and motivation

Bosonic systems appear in many natural and engineered settings e.g. photonics, circuit QED, trapped ions, cold atoms, ...

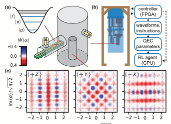


They are described by **infinite dimensional Hilbert spaces**.
This gives rise to unique phenomena (and opportunities):

Ong-Ou-Mandel effect [1]



Bosonic error correction [2]



Also: gives rise to **mathematical complications** (unbounded operators, continuous spectra, ...)

Often, simplified, **effective descriptions** are used.
This work answers the fundamental question :

Can we trust such effective descriptions to capture physical effects? **Yes!**

Effective descriptions of bosonic systems

Effective state space

Truncating the infinite-dimensional Hilbert space to a finite *effective dimension*
e.g. motivated by energy bounds

Common to simulate bosonic systems on classical (and DV quantum) computers

Effective dynamics

Restricting to specific families of *effective Hamiltonians*

Polynomial Hamiltonians - ubiquitous in the physics of bosons

What could go wrong ?

Bosonic CCR are impossible to satisfy in finite dimensions (even approximately)

$$[\hat q, \hat p] = i\hat I$$

The set of quantum correlations differ in the finite and infinite-dimensional cases [3]

Not all bosonic systems have native polynomial Hamiltonians

$$\cos\hat q \neq \sum_k \frac{\hat q^{2k}}{(2k)!}$$

Polynomial Hamiltonians could generate a restricted set of unitary evolutions: is CV universality even well-defined?

Continuous-variable quantum computing

Lloyd&Braunstein [4] : « a continuous-variable **(CV) universal quantum computer** should approximate any **polynomial hamiltonians** »

$$\hat H = \mathrm{poly}(\hat q_1, \hat p_1, \dots, \hat q_m, \hat p_m)$$

Universal set : $\{e^{i\frac{\pi}{2}(\hat q^2+\hat p^2)}, e^{ir\hat q_1\hat q_2}, e^{i\gamma\hat q^3}, e^{it\hat q}\}$

$$e^{iA\delta t}e^{iB\delta t}e^{-iA\delta t}e^{-iB\delta t} = e^{(AB-BA)\delta t^2} + \boxed{O(\delta t^3)}$$ Potentially unbounded

Natural definition but hard to answer questions such as

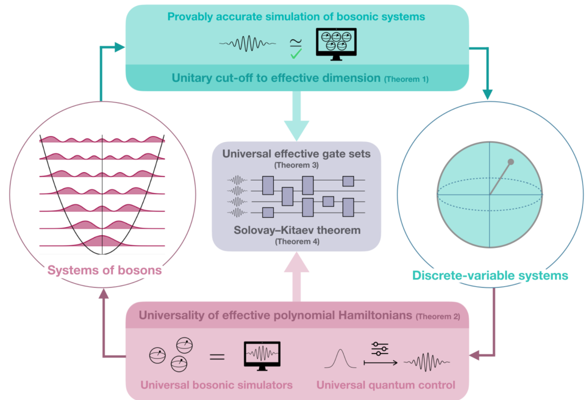
"How many qubits are needed to simulate the dynamics of a system of bosons to a given accuracy?"
(quantum simulation)

"Are different models of bosonic quantum computers equally powerful?"
(quantum complexity theory)

"Can a universal bosonic computer prepare an arbitrary quantum state?"
(quantum control)

All answered by our work!

Main results



Theorem 1: unitary truncation

Any infinite-dimensional ***physical* unitary evolution** with finite

$$E_{\hat U}(N) := \sup_{|\psi\rangle\in\mathcal{H}_N} \langle\psi|\hat U^\dagger \hat n \hat U|\psi\rangle$$

can be **rigorously approximated** to arbitrary precision by a **sequence of finite-dimensional ones**

$$d = \mathcal{O}\left(\frac{E}{\epsilon^4}E_{\hat U}\left(\frac{64E}{\epsilon^2}\right)\right)$$
$$\Downarrow$$
$$\exists \hat V_d \in \mathcal{U}(\mathcal{H}_d), \forall \hat V, \|\hat U - \hat V_d \oplus \hat V\|_\diamond^E \leq \epsilon$$

$$\mathcal{H} = \mathrm{span}\{|n\rangle\}_{n\in\mathbb{N}}$$
$$\mathcal{H}_N = \mathrm{span}\{|n\rangle\}_{0\leq n\leq N}$$
$$\hat n|n\rangle = n|n\rangle$$
$$\|\mathcal{E}\|_\diamond^E := \sup_{\mathrm{Tr}[\rho(\hat n\otimes\hat I)]\leq E} \|(\mathcal{E}\otimes\mathrm{id})\rho\|_1$$
$$\hat V_d \oplus \hat V = \begin{pmatrix} V_d & (0) \\ (0) & V \end{pmatrix}$$

Constructive proof: explicit finite-dimensional approximation (truncation + unitary rounding)

⇨ Rigorous upper **bound on the effective dimension** necessary to simulate bosonic dynamics

Theorem 2: DV universality of polynomial hamiltonians

Any finite-dimensional unitary evolution can be generated exactly by a polynomial Hamiltonian

For all Hermitian operator $\hat A$ on $\mathcal{H}_d$

$$\exists P_{\hat A} \in \mathbb{C}_{3d}[X,Y], \exists \hat A',\ P_{\hat A}(\hat q, \hat p) = \hat A \oplus \hat A'$$

and

$$e^{iP_{\hat A}(\hat q,\hat p)} = e^{i\hat A} \text{ on } \mathcal{H}_d$$

Constructive proof:
explicit polynomial Hamiltonian

$$P_{|n\rangle\langle n|}(\hat n) = \prod_{\substack{k=0\\k\neq n}}^{N} \frac{\hat n - k}{n-k}$$

$$P_{|n\rangle\langle n+k|}(\hat q,\hat p) = \sqrt{\frac{n!}{(n+k)!}} P_{|n\rangle\langle n|}(\hat n)\hat a^k$$

⇨ Universal bosonic simulators $\hat A \mapsto P_{\hat A}(\hat q, \hat p)$

⇨ Universal controllability/state engineering $e^{i\hat P_\psi(\hat q,\hat p)}|0\rangle \simeq |\psi\rangle$

CV universality of polynomial hamiltonians

Any infinite-dimensional ***physical* unitary** evolution can be **rigorously approximated** to arbitrary precision by a **sequence of unitary gates** generated by **polynomial Hamiltonians**

Start from infinite-dimensional physical unitary evolution

Theorem 1 ⇨ Find finite dimensional approximation

Theorem 2 ⇨ Reproduce exactly with polynomial hamiltonian

⇨ The definiton of universal CV quantum computer is well posed!
(the class of unitaries generated by polynomial hamiltonians is not restricted)

Solovay-Kitaev theorem

Infinite-dimensional physical unitary evolution

$\wr\wr$ Unitary truncation

Finite-dimensional unitary evolution

$\wr\wr$ DV Solovay-Kitaev theorem

Short sequence of finite-dimensional unitary gates

$||$ DV universality of polynomial Hamiltonians

Short sequence of infinite-dimensional unitary gates generated by polynomial Hamiltonians

Any infinite-dimensional ***physical* unitary** evolution can be **rigorously approximated** to arbitrary precision by a ***short* sequence of unitary gates** generated by **polynomial Hamiltonians**

⇨ any choice of universal gate sets over $\mathcal{H}_d$ for all d gives rise to a CV universal set of unitary gates generated by polynomial Hamiltonians, satisfying the CV Solovay-Kitaev theorem

Full paper :
F. Arzani, R. I. Booth, U. Chabaud
arxiv:2501.13857

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