Complex zeros of the wavefunction as theoretical and experimental signatures of non-Gaussianity

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Certifying a quantum resource without tomography

- Bosonic / continuous-variable systems: photonics, cavities, trapped ions, optomechanics
- The resource: **non-Gaussianity**, necessary for any bosonic computational advantage

> **quantum non-Gaussianity** = outside the closed convex hull of Gaussian states
>
> not just 'the Wigner function is not a Gaussian': mixtures are ruled out too

Related previous work

- tomographically complete measurements [Genoni '13, Hughes '14, Chabaud '21]
- or a non-Gaussian measurement [Filip & Mišta '11, Chabaud '21]
- single quadrature, but no guarantees [Park & Lu '17]

What is the simplest measurement that can witness it?

This talk: **a single quadrature.** One fixed homodyne setup.

Outline

0. Continuous-variable primer

modes, Wigner function, Husimi Q function, Gaussian states

1. A simple example: single photon

its quadrature distribution has a zero → use it as a witness

2. Which states have a zero in position distribution?

when the wave function is holomorphic: complex zeros count the stellar rank, phase shifts make them real

3. Simple example revisited: witnesses with formal guarantees

soundness, completeness, sample complexity, and realistic noise

Bosonic modes, in one slide

- One mode = a quantum harmonic oscillator, **dim H = ∞**

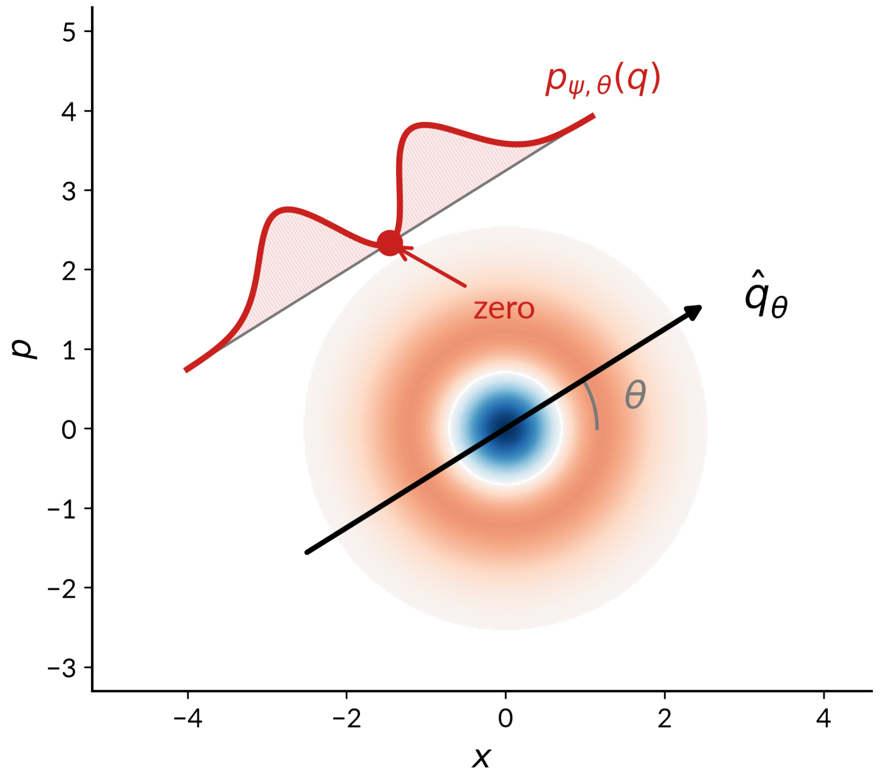
$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}), \qquad \hat{n} = \hat{a}^\dagger \hat{a}, \qquad \hat{n}\,|n\rangle = n\,|n\rangle$$

- Position-like observables at every angle:

$$\hat{q}_\theta = e^{-i\theta\hat{n}}\,\hat{x}\,e^{i\theta\hat{n}} = \hat{x}\cos\theta + \hat{p}\sin\theta$$

- **Homodyne detection** samples one of them
  - one angle = one fixed, standard optical setup

$$\text{homodyne at angle } \theta : \qquad p_{\psi,\theta}(q) = \left|\langle q|\psi\rangle_{\hat{q}_\theta}\right|^2$$



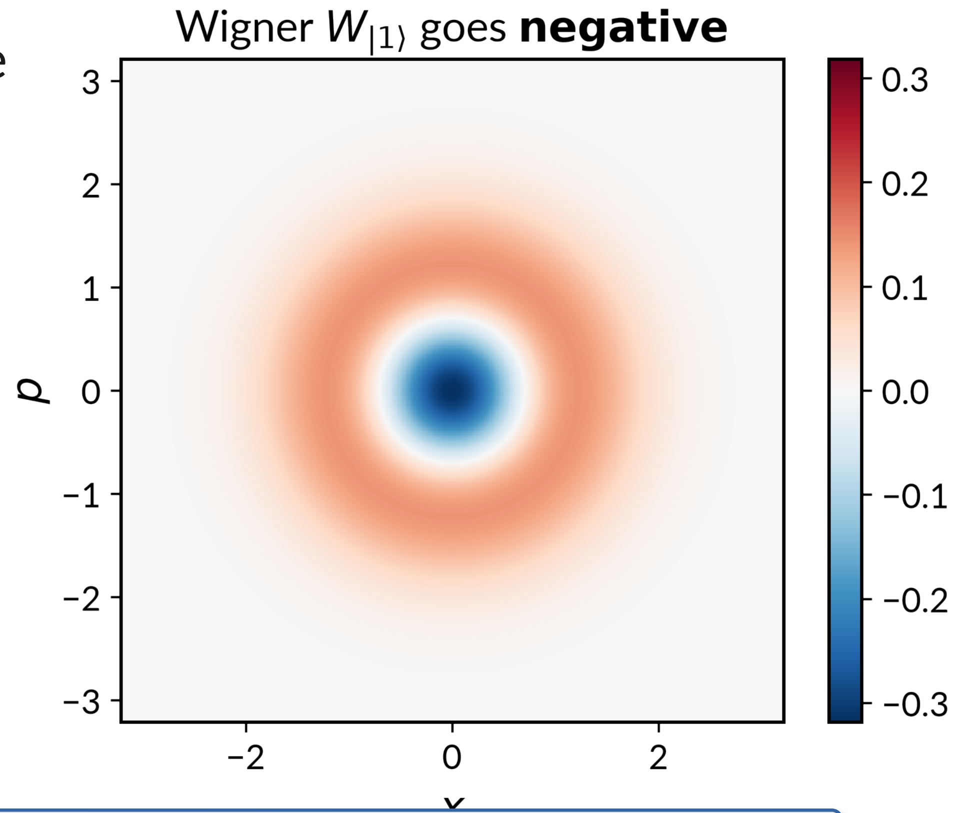
Phase space: the Wigner function

- Wigner function **W**: a quasi-probability on phase space

$$\int_{\mathbb{R}} W_\psi(q_\theta, p_\theta)\, \mathrm{d}p_\theta = p_{\psi,\theta}(q_\theta)$$

- its marginals are exactly the quadrature distributions

- *Quasi*, because it can go **negative**. It does for the single photon, at the origin.



a Gaussian state, three equivalent ways to say it

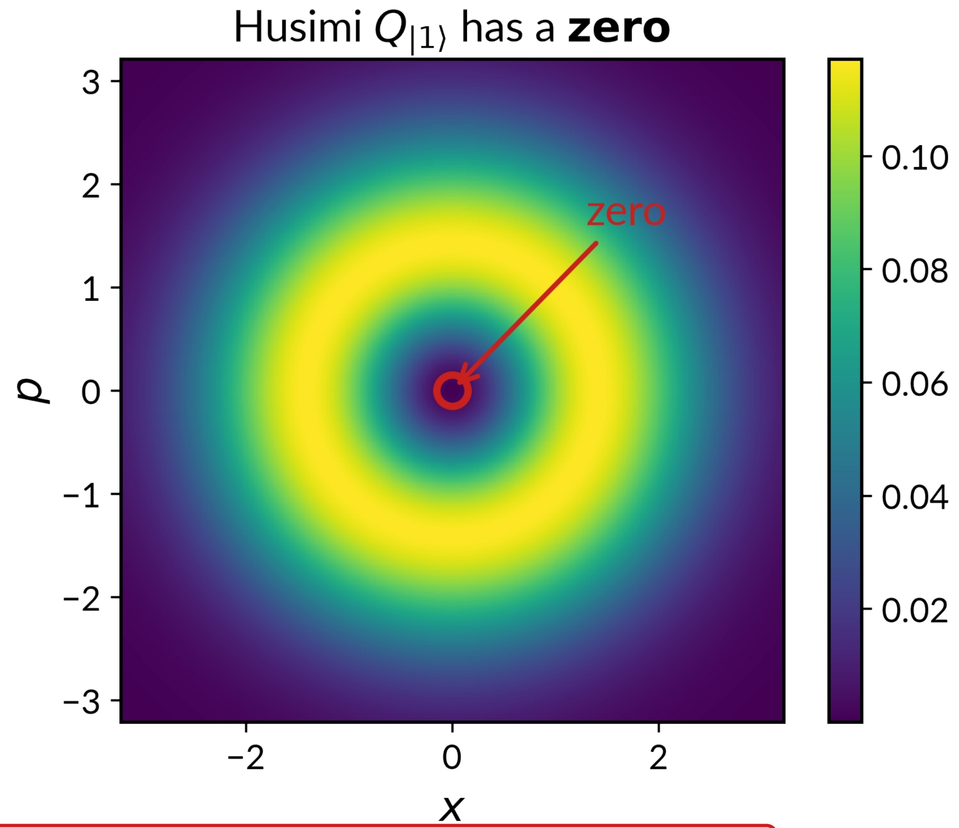
$$|\psi\rangle \text{ Gaussian} \iff W_\psi \text{ Gaussian} \iff p_{\psi,\theta} \text{ Gaussian for } \textit{every } \theta$$

The Husimi Q function, and Hudson's theorem

- Smooth **W** and you get **Q**, the coherent-state overlap

$$Q_\psi(\alpha) = \frac{1}{\pi}\left|\langle\alpha|\psi\rangle\right|^2 \qquad \text{(overlap with a coherent state)}$$

- **Q** is never negative, but it can **vanish**.
- For the single photon: exactly one zero, at the origin



Hudson's theorem (pure states)

$$|\psi\rangle \text{ non-Gaussian} \iff W_\psi < 0 \text{ somewhere} \iff Q_\psi \text{ has a zero}$$

How non-Gaussian? The stellar rank

$$F^\star_\psi(z) = \sum_{n\geq 0} \frac{\psi_n}{\sqrt{n!}} z^n, \qquad r^\star(\psi) = \#\{\text{zeros of } F^\star_\psi\}$$

$$Q_\psi(\alpha) = \frac{e^{-|\alpha|^2}}{\pi} \left|F^\star_\psi(\alpha^*)\right|^2 \qquad \Longrightarrow \qquad \text{zeros of } Q = \text{zeros of } F^\star$$

- **r*(|n⟩) = n, and r* = 0 $\iff$ Gaussian**

$$|\psi_r\rangle = \hat{D}(\alpha)\hat{S}(\chi)\sum_{n=0}^{r} c_n |n\rangle \qquad (r \text{ photon additions on a Gaussian})$$

- an operational ladder of non-Gaussianity, and it bounds computational usefulness [Chabaud & Walschaers '23]

All of this is normally read off **after tomography**, homodyne at ***every*** angle.

and it is exponentially costly [Mele et al., arXiv:2405.01431]

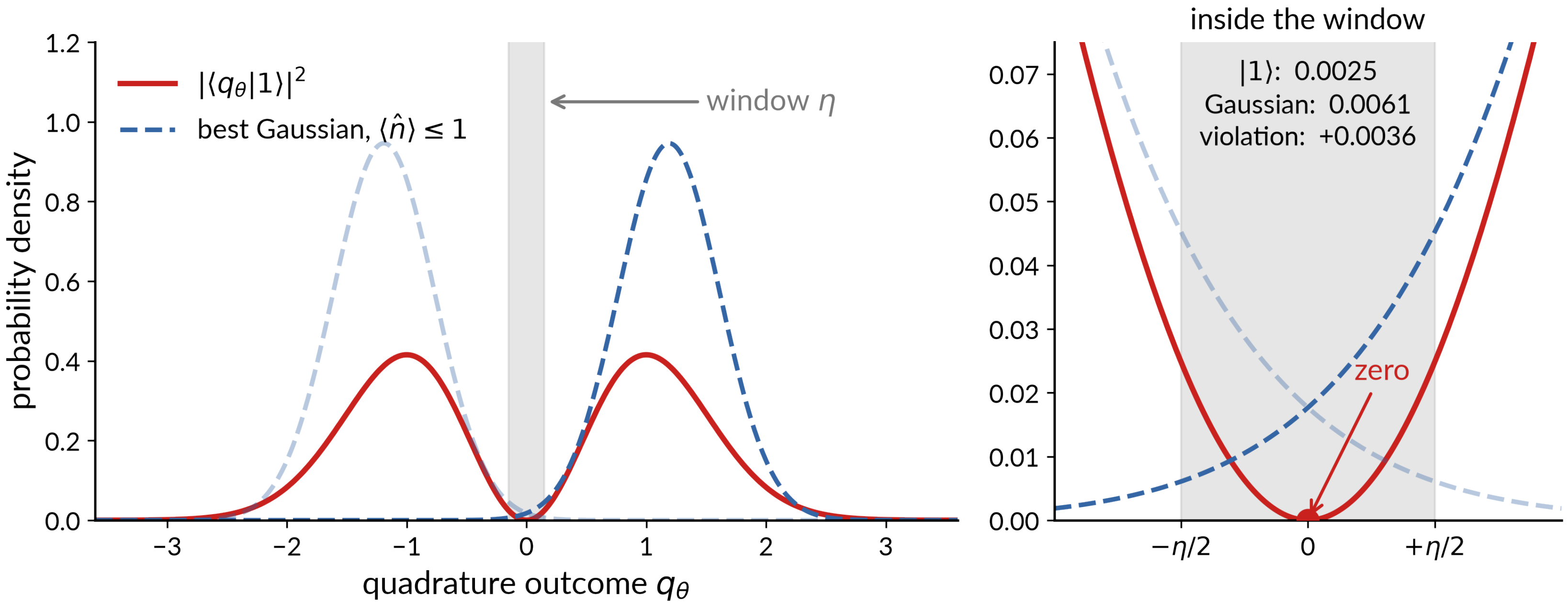
How many angles do we actually need?

Start from the simplest non-Gaussian state

$$\left|\langle q_\theta|1\rangle\right|^2 = \frac{2}{\sqrt{\pi}}\, q_\theta^2\, e^{-q_\theta^2} \xrightarrow{q_\theta=0} 0 \qquad \text{for every } \theta$$

- $|1\rangle$ is rotation-symmetric: **the same zero in every quadrature**

$$\left|\langle q_\theta|\alpha,\chi\rangle\right|^2 = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(q_\theta-\mu)^2}{2\sigma^2}} > 0 \quad \text{for every } q_\theta$$



A Gaussian can only *dodge* the window: it can never vanish inside it.

A zero is a signature of non-Gaussianity!

▲◧⬠⬡

- Count how often the outcome lands in a small window around the zero:

$$\hat{W}_{\theta,x,\eta} := \int_{x-\frac{\eta}{2}}^{x+\frac{\eta}{2}} |q\rangle\langle q|_{\hat{q}_\theta} \, \mathrm{d}q$$

|1⟩: 0.0025 **best Gaussian with ⟨n̂⟩ ≤ 1: 0.0061**

η = 0.3, window centered on the zero

- **But: without an energy bound this is empty.**

$$\hat{\rho}_{\text{cheat}} = \int p_{\hat{\rho},\theta}(q) \; |q\rangle\langle q|_{\hat{q}_\theta} \, \mathrm{d}q \qquad \text{same statistics, Gaussian mixture, } \langle \hat{n} \rangle = \infty$$

- Infinitely squeezed states are unphysical ⇒ promise **⟨n̂⟩ ≤ E** and the loophole closes
  - the conclusion is always "non-Gaussian, or more energetic than E"

So |1⟩ works. Which other states can this possibly certify?

Two questions

Q1 Which states have a zero in some quadrature?

a real zero is what a homodyne detector could actually see arXiv:2507.23468 Cerf et al.

Q2 Is the test sound, complete, and how many samples?

a witness is only useful with a computable threshold and error bars arXiv:2507.23005 Wassner et al.

Part 2 answers Q1, and it turns into a story about complex analysis.

The wavefunction is secretly holomorphic

- Assume only a mild energy condition:

$$\exists\, s > 1 : \qquad \langle s^{\hat{n}} \rangle_\psi = \sum_{n \geq 0} s^n |\psi_n|^2 < +\infty$$

  - satisfied by every state of finite stellar rank, a dense set
  - invariant under phase shifts, so it holds at every angle

Theorem (entire wavefunctions)

ψ extends from the real line to an **entire function on ℂ** of order at most 2.

- Hadamard–Weierstrass: such a function is **determined by its zeros**, up to a Gaussian factor

$$\psi(z) = e^{az^2+bz+c} \prod_k \left(1 - \frac{z}{\lambda_k}\right) e^{z/\lambda_k}$$

- Finitely many zeros, and the product collapses to a **polynomial**.

The hypothesis is necessary: $\psi(x) \propto \exp(-x^4)$ is non-Gaussian and has no zero. It violates the bound for every $s > 1$.

Hudson's theorem, moved onto the wavefunction

Theorem (Hudson for the wavefunction)

$$|\psi\rangle \text{ non-Gaussian} \iff \exists\, z \in \mathbb{C} : \ \psi(z) = 0$$

- And the count is exact: **number of complex zeros = stellar rank**

$$\psi(x) = P(x)\, e^{-ax^2+bx+c}, \qquad \deg P = r = r^\star(\psi), \qquad \Re(a) > 0$$

- Intuition: the rank is imprinted in the **wavefunction itself**, not only in Q or W
  - polynomial × Gaussian: the non-Gaussian content is the polynomial factor

A complex zero is not something a detector can see.

Homodyne only ever probes the real axis.

Phase shifts move the zeros

- Apply a phase shift **R̂(θ) = e^(−iθn̂)**, free evolution, and watch the zeros.

$$\psi(z,t) = \underbrace{e^{a(t)z^2+b(t)z+c(t)}}_{\text{Gaussian part}} \prod_{k=1}^{r} \big(z - \lambda_k(t)\big)$$

- Under any Gaussian evolution the envelope and the zeros **decouple**, coupled only through initial conditions

$$\ddot{\lambda}_k = (C^2 - 4AB)\,\lambda_k + CE - 2BD + 8B^2 \sum_{m\neq k} \frac{1}{(\lambda_k - \lambda_m)^3}$$

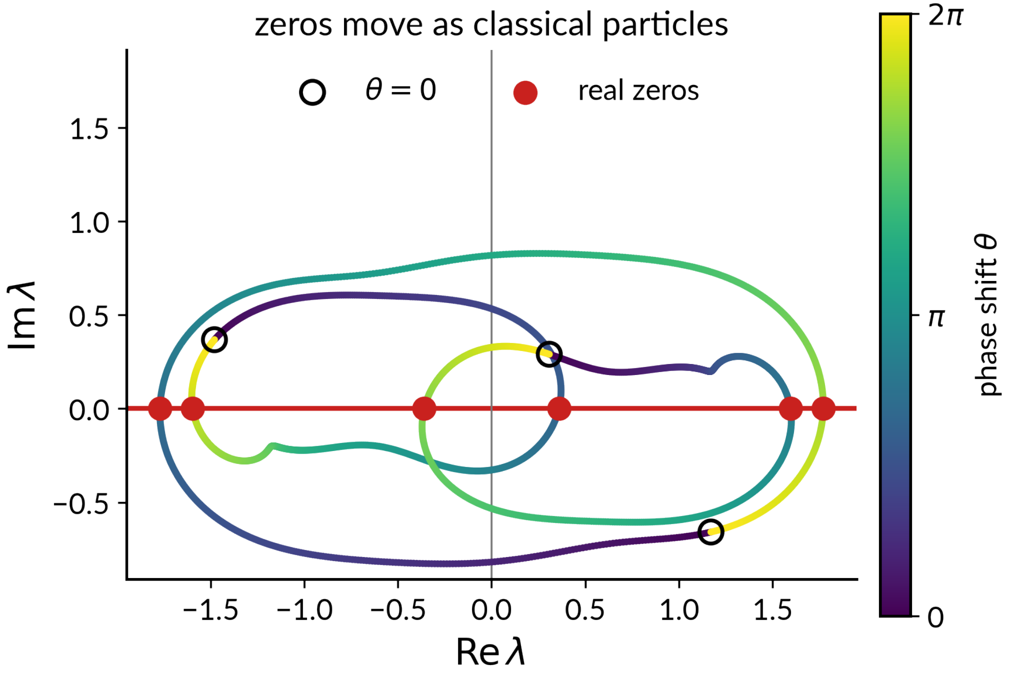
The zeros move like **classical particles**: an integrable Calogero–Moser system

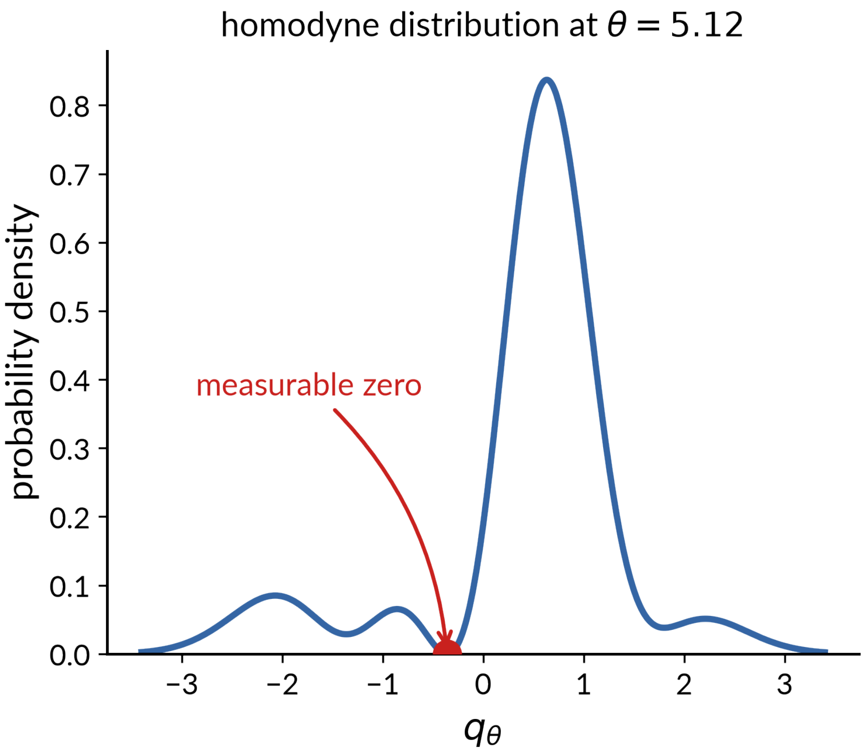
harmonic confinement + inverse-cube repulsion, solved by a matrix obeying $\ddot{X} + \omega^2 X = 0$

So: can we steer a zero onto the real axis?

Yes, always!

$$|\psi\rangle = \frac{1}{\sqrt{3}}\big(|0\rangle + i\,|1\rangle + |3\rangle\big), \qquad r^\star = 3$$





Theorem (Hudson theorem for finite stellar rank)

$$|\psi\rangle \;\; \text{non-Gaussian} \iff \exists\,\theta, x \in \mathbb{R}: \;\; \psi_\theta(x) = 0$$

Quantitative version (backup): if the zeros start far enough apart, the phase-shifted wavefunctions carry at least **2r** real zeros in total.

Two questions

Q1 Which states have a zero in some quadrature? ✓

a real zero is what a homodyne detector could actually see arXiv:2507.23468 Cerf et al.

Q2 Is the test sound, complete, and how many samples?

a witness is only useful with a computable threshold and error bars arXiv:2507.23005 Wassner et al.

Every non-Gaussian state of finite stellar rank has a measurable zero.

Now: is the test actually sound, and how many samples does it cost?

A witness with guarantees

- The states we are allowed to compete against:
  $\mathcal{S}^E := \mathrm{cch}\big(\{|\psi\rangle \ : \ \langle \hat{n}\rangle_\psi \le E\}\big), \qquad \mathcal{S}^E_r = \mathcal{S}^E \cap \{r^\star \le r\}$
  - mixed states too: no spectral component may exceed energy E
- The witness, and the best an energy-bounded Gaussian can do:

$$\hat{W}_{\theta,x,\eta} := \int_{x-\frac{\eta}{2}}^{x+\frac{\eta}{2}} |q\rangle\langle q|_{\hat{q}_\theta} \,\mathrm{d}q \qquad\qquad w^E_{\theta,x,\eta} := \inf_{\hat{\sigma}\in\mathcal{S}^E_0} \mathrm{Tr}\big(\hat{\sigma}\hat{W}_{\theta,x,\eta}\big)$$

the certification statement

$$\mathrm{Tr}\big(\hat{\rho}\hat{W}_{\theta,x,\eta}\big) < w^E_{\theta,x,\eta} \quad \Longrightarrow \quad \hat{\rho} \text{ non-Gaussian} \quad (\text{or } \hat{\rho} \notin \mathcal{S}^E)$$

Computing the threshold is a two-parameter numerical minimisation: the witness is linear, so the optimum is a pure Gaussian at maximal energy.

Sound and complete, for a small enough window

$$\textbf{Sound:}\quad \exists\,\delta>0:\ \forall\,0<\eta\le\delta,\qquad w^E_{\theta,x,\eta}>0$$

The threshold is strictly positive for Gaussians, and is ~linear in the window width

$$\textbf{Complete:}\quad p_{\hat\rho,\theta}(x)=0\implies \exists\,\delta>0:\ \forall\,\eta\le\delta,\quad \mathrm{Tr}\big(\hat\rho\hat W_{\theta,x,\eta}\big)<w^E_{\theta,x,\eta}$$

if the target really has a zero there, the violation is guaranteed to show up

- Why both hold: the two sides scale differently in the window size:

$$\underbrace{w^E_{\theta,x,\eta}\ \gtrsim\ c_1\,\eta}_{\text{Gaussian floor}}\qquad \text{vs}\qquad \underbrace{\mathrm{Tr}\big(\hat\rho\hat W_{\theta,x,\eta}\big)\ \lesssim\ c_2\,\eta^2}_{\text{quadratic near a zero}}$$

η^2 beats η once η is small enough.

$$\mathcal{S}^E_r \text{ is compact in the trace norm}\implies \text{the infimum is a minimum}$$

the technical crux, and of independent interest: it makes the infimum attained, so the threshold is a value some Gaussian actually achieves

The protocol

1. Find an angle θ and a point x where the distribution has a zero
   from theory, or from a first batch of data
2. Fix a window η and build the witness
   a workable η is guaranteed to exist
3. Compute the threshold
   numerical minimisation over energy-bounded Gaussians, done offline
4. Measure that one quadrature, count outcomes in the window
5. Compare
   below threshold ⇒ non-Gaussian (or over energy). Above ⇒ no conclusion; shrink η and retry

Δ = the violation: threshold minus what you measured

$$\langle\hat{\Delta}^{E}_{\theta,x,\eta}\rangle = w^{E}_{\theta,x,\eta} - \mathrm{Tr}\big(\hat{\rho}\hat{W}_{\theta,x,\eta}\big) \;\le\; \inf_{\hat{\sigma}\in\mathcal{S}^{E}_{0}} D(\hat{\rho},\hat{\sigma})$$

It lower-bounds the trace distance from the prepared state to every energy-bounded Gaussian, so the size of the violation is itself meaningful.

How many samples?

- The estimator is very simple:

$$\bar{w} = \frac{1}{M}\sum_{j=1}^{M} \chi_{[x-\frac{\eta}{2},\, x+\frac{\eta}{2}]}(q_j) \qquad \text{(just count the samples in the window)}$$

- i.i.d. bounded variables → Hoeffding:

$$p_{\text{fail}} \leq e^{-2M\epsilon^2} \qquad \Longleftrightarrow \qquad M \geq \frac{\log(1/\delta)}{2\epsilon^2} \quad \text{for confidence } 1-\delta$$

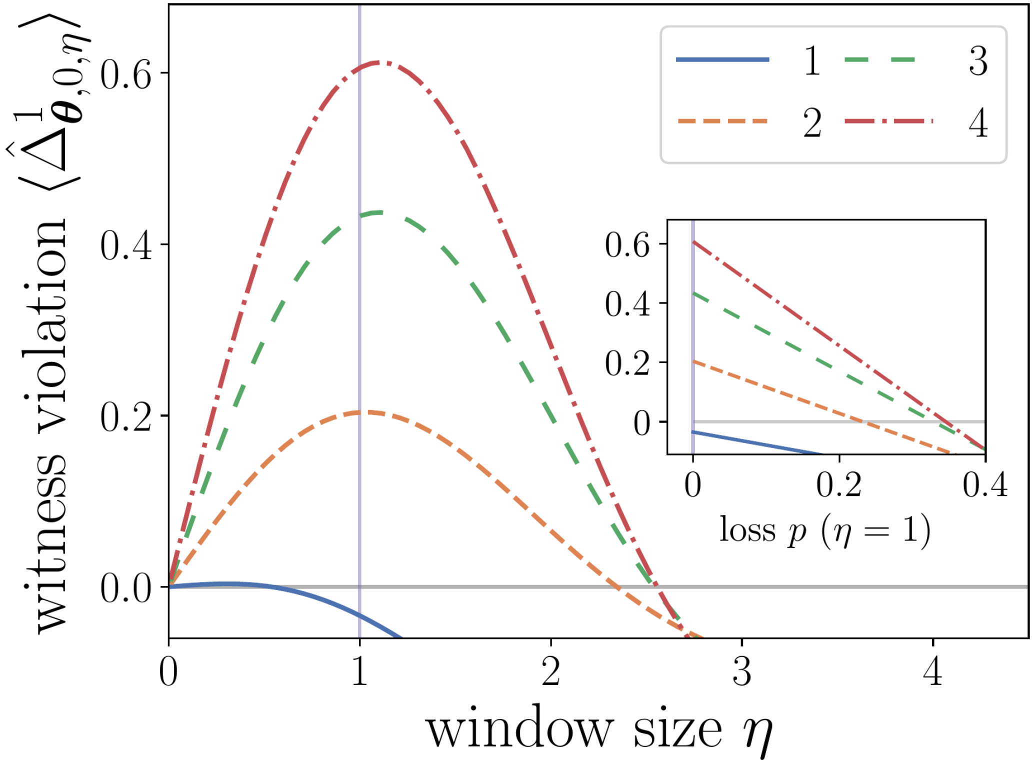
$$\text{declare non-Gaussian when} \quad \bar{w} < w^{E}_{\theta,x,\eta} - \epsilon, \qquad \epsilon = \text{margin below the Gaussian floor}$$

This also closes a loophole.

A cheater could centre the window far from where any outcome lands, but then the estimate is **below threshold only with vanishing probability**, so the confidence never arrives.

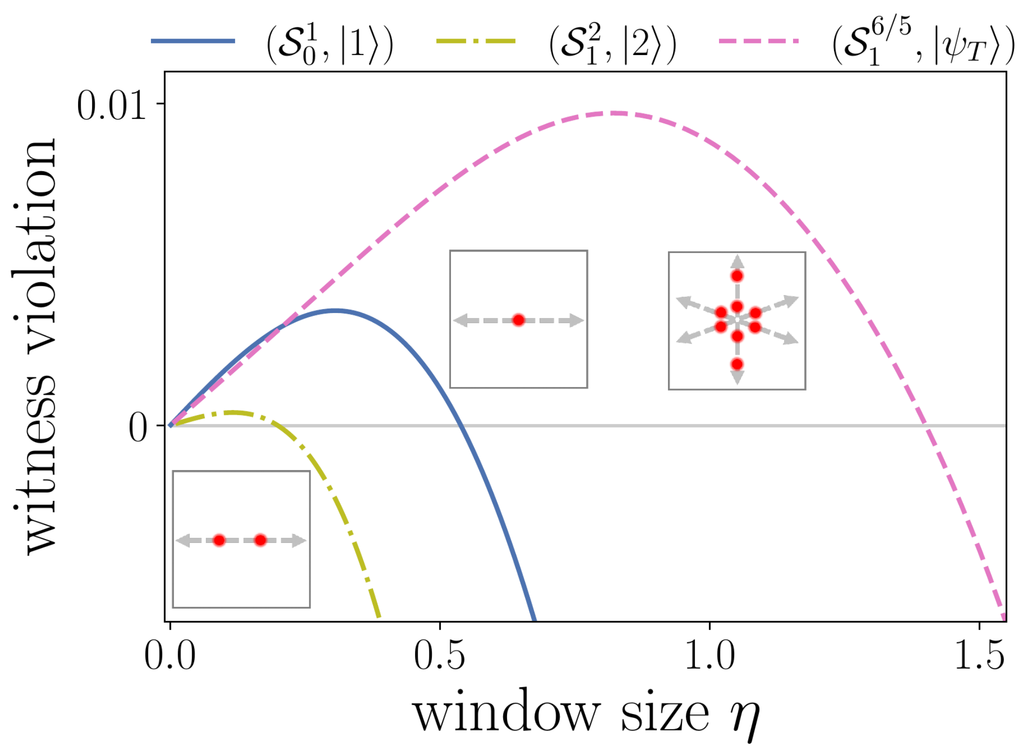
- Multiple angles: the same argument with a **union bound** over the angles

But is one quadrature enough in practice?



- Target $|1\rangle$, **one** quadrature: the violation is real but **tiny**, and dies at ~2% loss
- Add **one more angle** and the violation grows by roughly **two orders of magnitude**
- Loss tolerance goes from ~2% to ~35% (inset)
- **Basically:** a single quadrature suffices in principle, and for near-perfect states. In the lab you will want more (still a long way from tomography)

Beyond $|1\rangle$: stellar rank, and states with many zeros

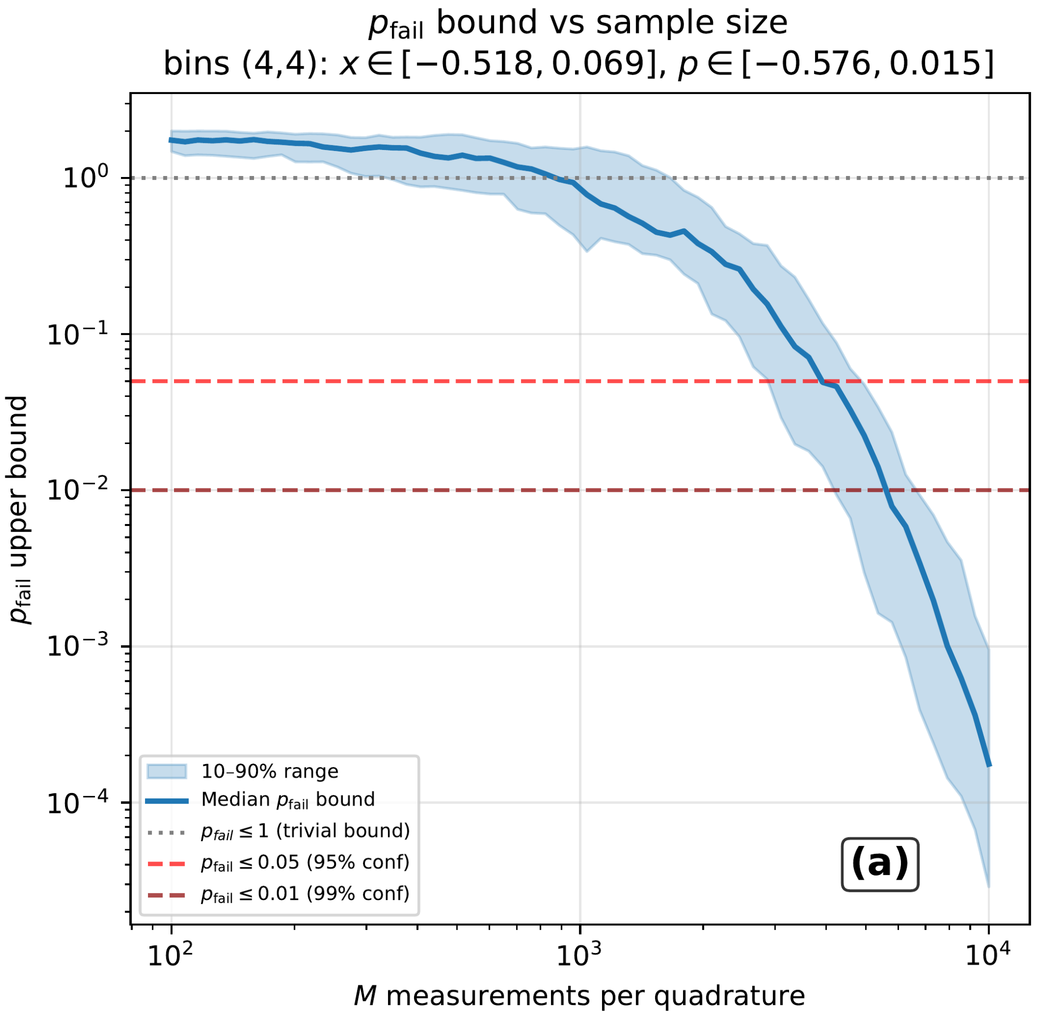


- Windows around k zeros, with the threshold optimised over all states of stellar rank below k, certify stellar rank k
- Robustness depends on a subtle trade-off: high rank, low energy, many zeros near the origin
- Cat states have **infinitely many real zeros** along almost every angle

$$\left|\langle q_\theta|\psi_\alpha\rangle\right|^2 \propto \cdots + 2e^{-q^2-2\Re(\alpha e^{-i\theta})^2}\cos\left(2\sqrt{2}\,\Im(\alpha e^{-i\theta})\,q\right)$$

⇒ **arbitrarily high stellar rank** witnessed from a single quadrature

End to end, on simulated homodyne data



- **Batch 1** builds the histogram and picks the window
- **Batch 2** evaluates the witness on fresh copies
- Splitting the data ensures no bias in the certificate

≈ 10^4 homodyne samples per quadrature for 99% confidence

- No full model of the state is needed, only that it has a zero somewhere in the measured range

$p_{\mathrm{fail}} = P(\text{measured violation AND it was a Gaussian of small } E)$

Conclusions

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- The zeros of the wavefunction are **signatures of non-Gaussianity**, and they count the stellar rank
- Under Gaussian evolution they move as an integrable **Calogero–Moser** system; phase shifts bring them onto the real axis
- Therefore, quantum non-Gaussianity, and arbitrary stellar rank, can be witnessed from **a single quadrature**: sound, complete, with sample complexity guarantees
- Open: which targets are most robust? mixed states? heterodyne version? infinite stellar rank?

Cerf et al. arXiv:2507.23468

Wassner et al. arXiv:2507.23005

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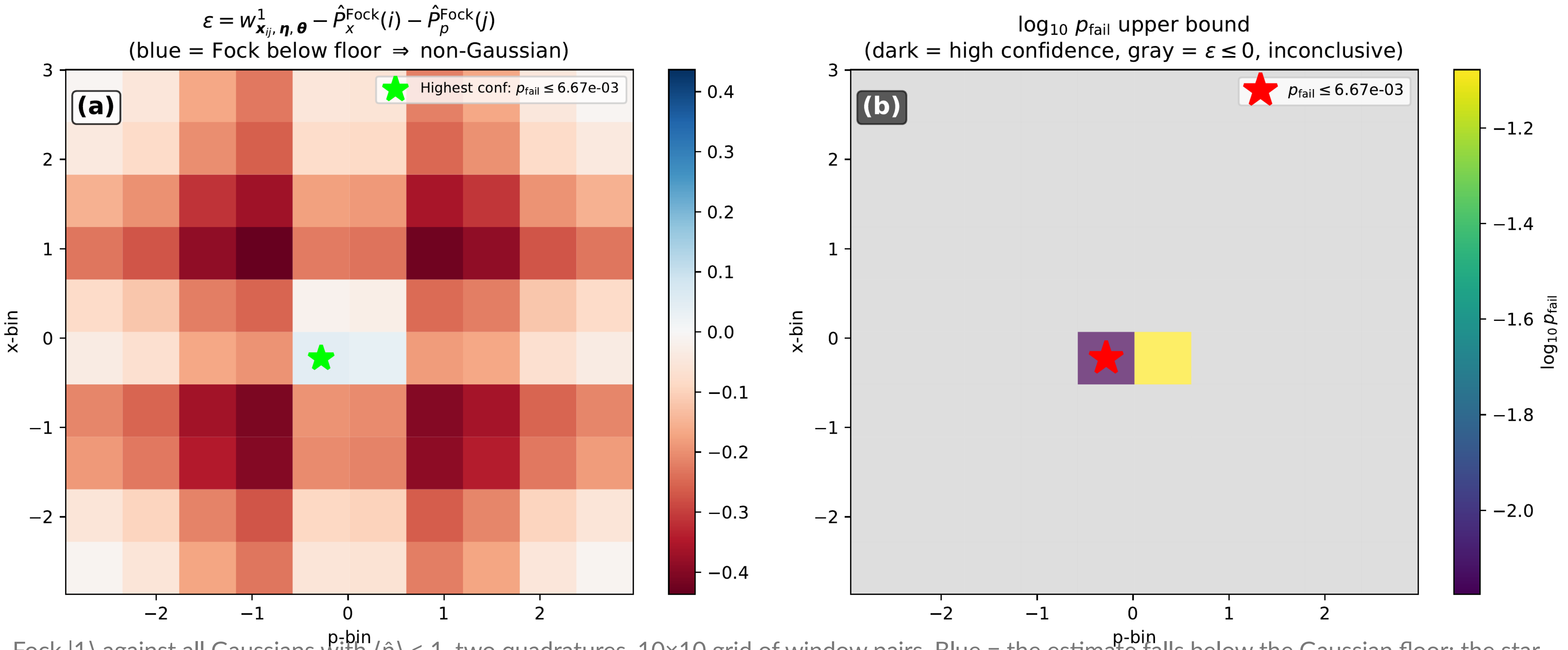
Thank you!

Appendix

Choosing the window from data

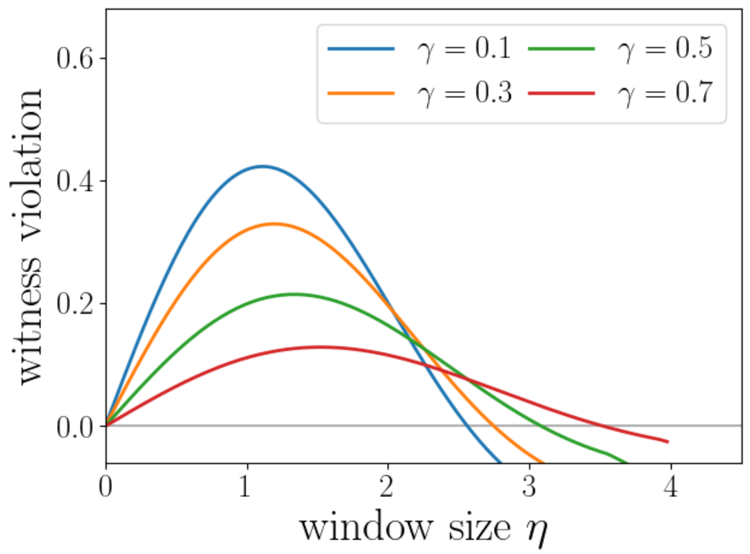


Fock |1⟩ against all Gaussians with ⟨n̂⟩ ≤ 1, two quadratures, 10×10 grid of window pairs. Blue = the estimate falls below the Gaussian floor; the star marks the best bound on p_fail (3.6×10^{-3} from 5000 samples per quadrature).

Soft (Gaussian-blurred) windows

$$\hat{V}_{\theta,x,\gamma} = \int g_{x,\gamma}(q)\,|q\rangle\langle q|_{\hat{q}_\theta}\,\mathrm{d}q, \qquad \hat{W}^{\mathrm{soft}}_{\theta,x,\gamma,\eta} = \int_{x-\frac{\eta}{2}}^{x+\frac{\eta}{2}} \hat{V}_{\theta,y,\gamma}\,\mathrm{d}y$$

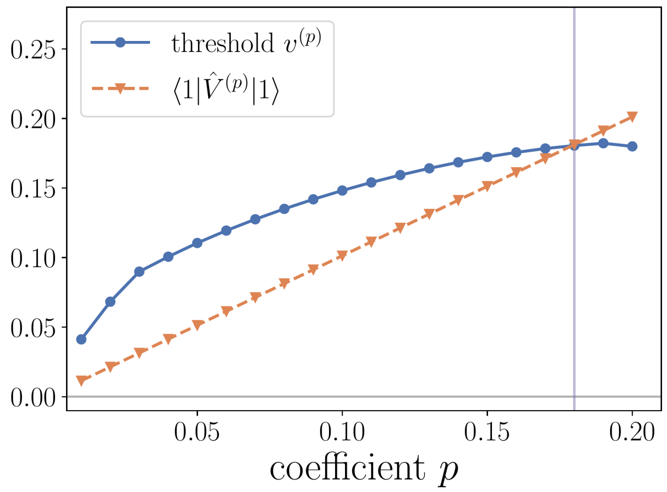
- γ plays the role of a finite homodyne resolution, added classical noise
- Closed-form threshold expressions survive the convolution



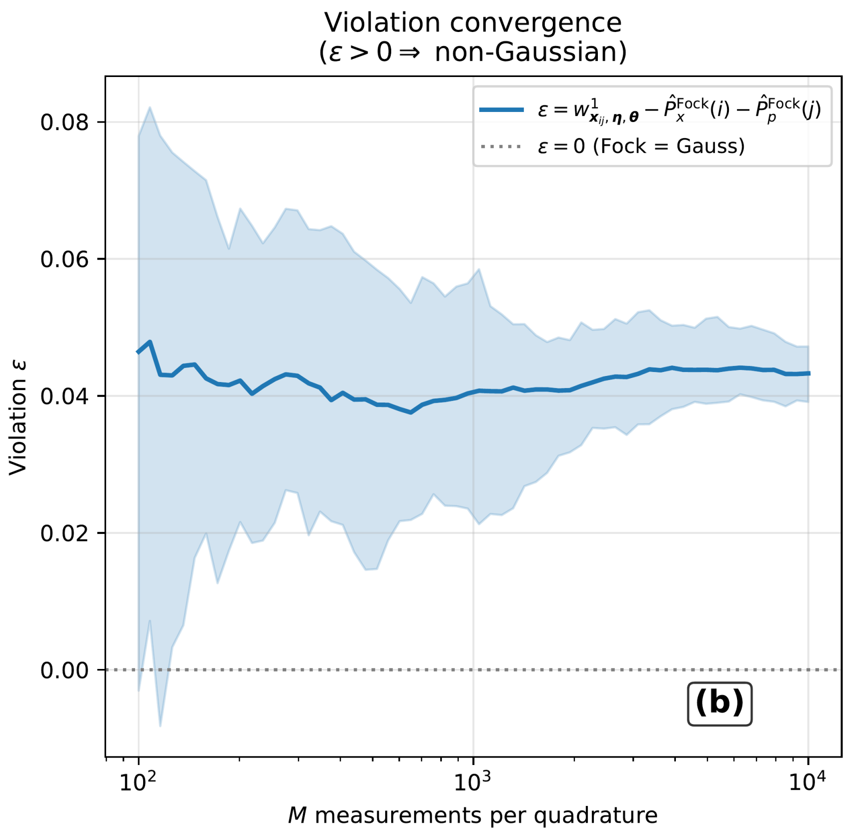
Dropping the energy promise

$$\hat{V}^{(p)}_{\boldsymbol{\theta},\boldsymbol{x},\eta} := p\,\hat{n} + (1-p)\,\hat{W}_{\boldsymbol{\theta},\boldsymbol{x},\eta}$$

- Convex mixture of the witness with the number operator: the adversary's energy now competes with its ability to dodge the window
- Optimisation becomes a minimal eigenvalue over a (k×k) matrix, no energy constraint needed
- Price: the energy term must be measured, so at least two conjugate quadratures



Convergence of the violation itself



Second panel of the run in the paper/main slides: the witness violation stabilises at a positive value as the number of samples grows, so the Fock state really does beat the Gaussian floor rather than fluctuating across it.

Quantitative version of Hudson theorem for finite SR

$$\min_{i,j}\left|\lambda_i(0)-\lambda_j(0)\right| \geq \sqrt{\frac{r-1}{a(0)}} \quad \Longrightarrow \quad \#\{\text{real zeros over all } \theta\} \;\geq\; 2r$$

- Proof idea: treat the Calogero–Moser interaction as a perturbation of simple harmonic motion, legitimate exactly when the zeros start far apart
- Each zero then crosses the real axis twice over a full 2π of phase shift
- Conjecture (numerics): the count is exactly 2r under that condition

$$\hat{W}_{\boldsymbol{\theta},\boldsymbol{x},\boldsymbol{\eta}} = \sum_{j=1}^{N} \hat{W}_{\theta_j,x_j,\eta_j}, \qquad p_{\text{fail}} \leq \sum_j e^{-2M_j(\epsilon/N)^2}$$

Multi-quadrature witness and its sample complexity (union bound + Hoeffding).